The computational complexity of graph contractions I: polynomially solvable and NP-complete cases *

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Abstract

For a fixed pattern graph H, let H-Contractibility denote the problem of deciding whether a given input graph is contractible to H. This paper is part I of our study on the computational complexity of the H-Contractibility problem. We continue a line of research that was started in 1987 by Brouwer & Veldman, and we determine the computational complexity of the H-Contractibility problem for certain classes of pattern graphs. In particular, we pin-point the complexity for all graphs H with five vertices except for two graphs, whose polynomial time algorithms are presented in part II. Interestingly, in all connected cases that are known to be polynomially solvable, the pattern graph H has a dominating vertex, whereas in all cases that are known to be NP-complete, the pattern graph H does not have a dominating vertex.

Keywords: graph, edge contraction, dominating vertex, computational complexity.

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1 Introduction

All graphs in this paper are undirected, finite, and *simple*, i.e., without loops and multiple edges. If no confusion is possible, we write $V = V_G$ and $E = E_G$ for a graph $G = (V_G, E_G)$. A graph G is a *subgraph* of a graph H, denoted by $G \subseteq H$, if $V_G \subseteq V_H$ and $E_G \subseteq E_H$. For a subset $U \subseteq V_G$ we denote by $G[U]$ the *induced subgraph* of G over U; hence $G[U] = (U, E_G \cap (U \times U))$. A graph G is called *connected* if for every pair of distinct vertices u and v, there exists a *path* connecting u and v, i.e., a sequence of distinct vertices starting by u and ending by v where each pair of consecutive vertices forms an edge of G. Each maximal connected subgraph of a graph G is called a *component* of G. A graph P_n denotes a path on n vertices. Two graphs G and $\tilde{G}$ are called *isomorphic*,

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denoted by $G \simeq \tilde{G}$, if there exists a one-to-one mapping f of vertices of G onto vertices of $\tilde{G}$ such that $[u, v] \in E_G$ if and only if $[f(u), f(v)] \in E_{\tilde{G}}$. The mapping f is called an *isomorphism* between G and $\tilde{G}$.

Let $G = (V, E)$ be a graph, and let $e = [u, v] \in E$ be an arbitrary edge with *end-vertices* u and v. We also say that end-vertices u and v are *adjacent*. The *edge contraction* of edge e in G removes the two end-vertices u and v from G, and replaces them by a new vertex that is adjacent to precisely those vertices to which u or v were adjacent. We denote the resulting graph by $G \backslash e$. The *edge deletion* of edge e removes e from E. The *edge subdivision* of e removes e from E, and introduces a new vertex that is adjacent to the two end-vertices u and v. A graph G is contractible to a graph H (graph G is H-contractible), if H can be obtained from G by a sequence of edge contractions. A graph G contains a graph H as a *minor*, if H can be obtained from G by a sequence of edge contractions and edge deletions. A graph G is a *subdivision* of a graph H, if G can be obtained from H by a sequence of edge subdivisions.

Now let $H = (V_H, E_H)$ be some fixed connected graph with vertex set $V_H = \{h_1, \ldots, h_k\}$. There is a number of natural and elementary algorithmic problems that check whether the structure of the graph H shows up as a *pattern* within the structure of some input graph G:

- H-MINOR CONTAINMENT
  Instance: A graph $G = (V, E)$.
  Question: Does G contain H as a minor?

- H-SUBDIVISION SUBGRAPH
  Instance: A graph $G = (V, E)$.
  Question: Does G contain a subgraph that is isomorphic to some subdivision of H?

- ANCHORED H-SUBDIVISION SUBGRAPH
  Instance: A graph $G = (V, E)$ and k pairwise distinct vertices $v_1, \ldots, v_k$ in V.
  Question: Does G contain a subgraph that is isomorphic to some subdivision of H, such that the isomorphism maps vertex v_i of the subgraph of G into vertex h_i of the subdivision of H, for $1 \leq i \leq k$?

- H-CONTRACTIBILITY
  Instance: A graph $G = (V, E)$.
  Question: Is G contractible to H?

A celebrated result by Robertson & Seymour [5] states that the H-MINOR CONTAINMENT problem can be solved in polynomial time for every *fixed* pattern graph H. In fact, [5] fully settles the complexity of the first three problems on our problem list above:

Proposition 1 ([5]) *For any fixed pattern graph H, the three problems H-*MINOR CONTAINMENT*, H-*SUBDIVISION SUBGRAPH*, and* ANCHORED H-SUBDIVISION SUBGRAPH *are polynomially solvable.*

What about the fourth problem on our list, H-CONTRACTIBILITY? Perhaps surprisingly, there exist pattern graphs H for which this problem is NP-complete to decide! For instance, Brouwer & Veldman [2] have shown that P_4-CONTRACTIBILITY is NP-complete. They make the following observation, which shows one only has to consider connected pattern graphs H.

Observation 1.1 ([2]) *The H-*CONTRACTIBILITY *problem is solvable in polynomial time if and and only if the H_i-*CONTRACTIBILITY *problem is solvable in polynomial time for every component H_i of H.*

A *cycle* C on n vertices is a graph whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_C = \{[v_1, v_2], \ldots, [v_{n-1}, v_n], [v_n, v_1]\}$. A graph C_n denotes a cycle on n vertices. A graph that does not contain a C_3 as a subgraph is said to be *triangle-free.* A graph $G = (V, E)$ that has a vertex u such that every edge in E is adjacent to u is called a *star.* The main result of [2] is the following.

Theorem 2 ([2]) *Let H be a connected triangle-free graph. The H-*CONTRACTIBILITY *problem is polynomially solvable if H is a star, and it is* NP*-complete otherwise.*

A *dominating* vertex is a vertex that is adjacent to all other vertices. Note that an equivalent way of stating Theorem 2 would be the following: The H-CONTRACTIBILITY problem is NP-complete for every connected triangle-free graph H without a dominating vertex. The H-CONTRACTIBILITY problem is polynomially solvable for every connected triangle-free graph H with a dominating vertex.

Moreover, in [2] the complexity of the H-CONTRACTIBILITY problem is determined for all 'small' connected pattern graphs H: For $H = P_4$ and $H = C_4$, the problem is NP-complete (as implied by Theorem 2). For every other pattern graph H on at most four vertices, the problem is polynomially solvable.

The exact separating line between polynomially solvable cases and NP-complete cases of this problem (under P≠NP) is unknown and unclear. Brouwer & Veldman [2] write at the end of their paper that they expect the class of polynomially solvable cases to be very limited.

Watanabe, Ae & Nakamura [6] consider remotely related edge contraction problems where the goal is to find the minimum number of edge contractions that transform a given input graph G into a pattern from a certain given pattern class.

We follow the line of research that has been initiated by Brouwer & Veldman [2], and we classify the complexity of the H-CONTRACTIBILITY problem for certain classes of pattern graphs that, in particular, contain all 'small' pattern graphs H with at most five vertices. Our results can be summarized as follows:

Theorem 3 *Let H be a connected graph on at most five vertices. If H has a dominating vertex, then the H-*CONTRACTIBILITY *problem is polynomially solvable. If H does not have a dominating vertex, then the H-*CONTRACTIBILITY *problem is* NP*-complete.*

It is difficult for us *not* to conjecture that the presence of a dominating vertex in the pattern graph H precisely separates the easy cases from the hard cases. However, we have no evidence for such a conjecture.

There are fifteen graphs H on five vertices that are not covered by Theorem 2; these are exactly the connected graphs on five vertices that do contain a triangle; see Figures 1 and 2 for pictures of all these graphs. It turned out that ten of these fifteen graphs yield polynomially solvable H-CONTRACTIBILITY problems, whereas the other five of them yield NP-complete problems. Many of our results are actually more general: They do not only provide a specialized result for one particular five-vertex graph, but they do provide a result for an infinite family of pattern graphs, from which the result on the five-vertex graph falls out as a special case. Our main contributions may be summarized as follows:

(1) We analyze a class of cases where H contains one, two, or three dominating vertices, and where the components of the subgraph induced by the set of non-dominating vertices are all paths (of possibly different length). In Section 3, we prove that three subfamilies of this class

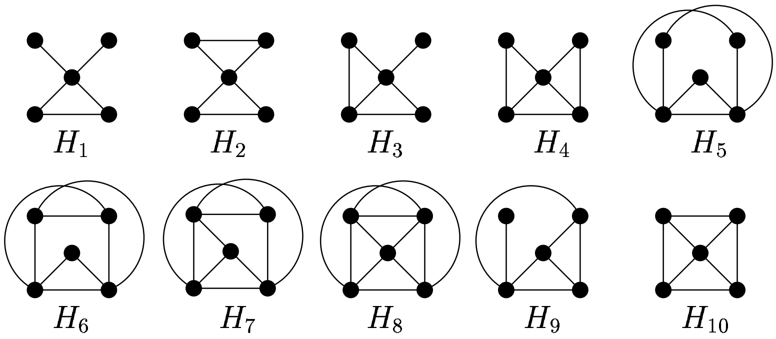

Figure 1: The graphs $H_1, H_2, \dots, H_{10}$.

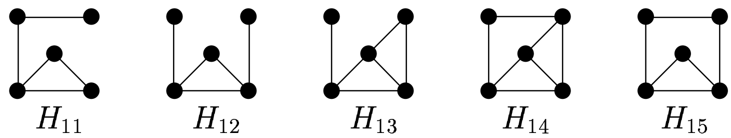

Figure 2: The graphs $H_{11}, H_{12}, \dots, H_{15}$.

yield polynomially solvable H-Contractibility problems. These classes contain the eight graphs $H_1, \dots, H_8$ on five vertices as depicted in Figure 1.

Our structural results show that in case *some* H-contraction exists, then there also exists an H-contraction of a fairly primitive form. In our algorithmic results, we then enumerate all possibilities for these primitive pieces, and settle the remaining problems by applying the results of Robertson & Seymour [5].

(2) For the two five-vertex graphs H_9 and H_{10} as shown in Figure 1, we were not able to find 'straightforward' polynomial time algorithms. Our algorithms are based on lengthy (!) combinatorial investigations of potential contractions of an input graph to H_9 and H_{10}. We do not include the algorithms in this paper due to the length of their proofs but refer to [4] for their presentation.

(3) In Section 4 we give a number of NP-completeness results. We present a generic NP-completeness construction. As a special case, this yields the NP-completeness of the H_{15}-Contractibility problem for the graph H_{15} in Figure 2. Moreover, we give four NP-completeness proofs for the four (five-vertex) graphs H_{11}, H_{12}, H_{13}, and H_{14} in Figure 2. All four proofs are done by reduction from Hypergraph 2-Colorability and they are inspired by a similar NP-completeness argument of Brouwer & Veldman [2].

2 Preliminaries

For graph terminology not defined below (or in the introduction) we refer to [1]. For a vertex u in a graph $G = (V, E)$ we denote its *neighborhood*, i.e., the set of adjacent vertices, by $N(u) = \{v \mid [u, v] \in E\}$. The *degree* of a vertex u is the number of edges incident with it, or equivalently the size of its neighborhood. The *neighborhood* $N(U)$ of a subset $U \subseteq V$ is defined as $\bigcup_{u \in U} N(u) \backslash U$, and we call the vertices in $N(U)$ *neighbors* of U. If $v \in N(U)$ for some subset $U \subseteq V$ we say that v is *adjacent* to U. Two subsets $U, U' \subseteq V$ with $U \cap U' = \emptyset$ are *adjacent*, if there exist vertices $u \in U$ and $u \in U'$ with $[u, u'] \in E$.

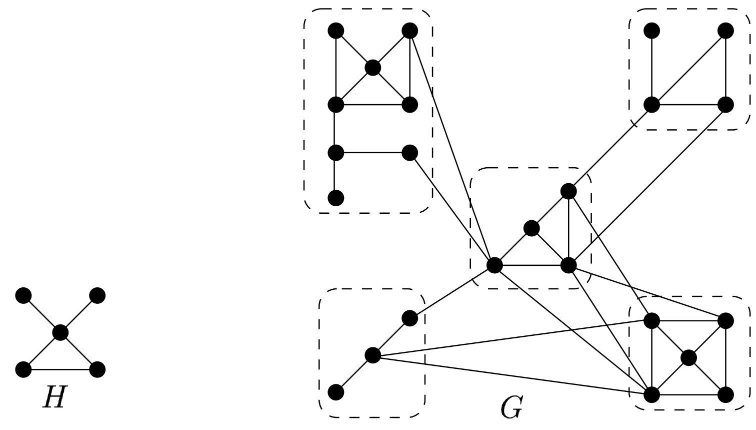

Figure 3: A graph H and an H-contractible graph G.

A graph $G = (V, E)$ is called *k-connected* if $G[V \backslash U]$ is connected for any set $U \subseteq V$ of at most $k-1$ vertices. A *k-vertex cut* is a subset $S \subseteq V$ of size k such that $G[V \backslash S]$ is not connected. The vertex in a 1-vertex cut of a graph G is called a *cutvertex.* Each maximal 2-connected subgraph of a graph G is called a *block* of G. Note that by their maximality any two blocks of G have at most one vertex (which is a cutvertex of G) in common.

Consider a graph $G = (V_G, E_G)$ that is contractible to a graph $H = (V_H, E_H)$. An equivalent (and for our purposes more convenient) way of stating this fact is that

- for every vertex h in V_H, there is a corresponding connected subset $W(h) \subseteq V$ of vertices in G (so the induced subgraph of G over $W(h)$ is connected); we call a set $W(h)$ an *H-witness set* of G for vertex h and call the set $\mathcal{W} = \{W(h) \mid h \in V_H\}$ an *H-witness structure* of G;

- for every edge $e = [h_1, h_2] \in E_H$, there is at least one edge in G that connects the vertex set $W(h_1)$ to the vertex set $W(h_2)$;

- for every two vertices h_1, h_2 in H that are not connected by an edge in E_H, there are no edges between $W(h_1)$ and $W(h_2)$.

If for every $h \in V_H$, we contract the vertices in $W(h)$ to a single vertex, then we end up with the graph H. See Figure 3 for an example. Note that in general, witness sets $W(h)$ are not uniquely defined (since there may be many different sequences of contractions that lead from G to H). In our polynomial time algorithms, we will explore the structure of the witness sets, and often prove that there exists *at least one* witness structure with certain 'strong' and 'nice' properties.

The following algorithmic problem plays a crucial role in our study on the H-CONTRACTIBILITY problem.

DISJOINT CONNECTED SUBGRAPHS(k)
Instance: A graph $G = (V, E)$ and non-empty subsets $Z_1, \ldots, Z_t \subseteq V$ such that $\sum_{i=1}^{t} |Z_i| \leq k$.
Question: Does G contain mutually disjoint connected subgraphs $G_i \subseteq G$ such that $Z_i \subseteq V_{G_i}$ for $1 \leq i \leq t$?

Robertson & Seymour [5] proved the following.

Theorem 4 ([5]) *The* DISJOINT CONNECTED SUBGRAPHS*(k) problem is solvable in polynomial time for all $k \geq 1$.*

A *complete* graph is a graph with an edge between every pair of vertices. The complete graph on p vertices is denoted by K_p. As an immediate consequence of Theorem 4 we can solve the following problem in polynomial time as well.

K_p-FIXED CONTRACTIBILITY
Instance: A graph $G = (V, E)$ and $t \leq p$ subsets $Z_1, \dots, Z_t \subseteq V$ such that $\sum_{i=1}^{t} |Z_i| \leq p$.
Question: Can G be contracted to K_p with K_p-witness sets $U_1, \dots U_p$ such that $Z_i \subseteq U_i$ for $1 \leq i \leq t$?

Corollary 5 *The K_p-FIXED CONTRACTIBILITY problem is solvable in polynomial time.*

Proof: Suppose we are given a graph $G = (V, E)$ and $t \leq p$ subsets $Z_1, \dots, Z_t \subseteq V$ such that $\sum_{i=1}^{t} |Z_i| \leq p$. Define $Z_i := \emptyset$ for $i = t+1, \dots, p$. Then we act as follows. First we guess $p(p-1)$ different edges. Denote these edges by $[u_i^j, u_j^i]$ for all $1 \leq i < j \leq p$, where we allow $u_h^j = u_i^j$ for some indices h, i, j. For $i = 1, \dots, p$ we define $Z'_i = Z_i \cup \{u_i^1, \dots, u_i^{i-1}, u_i^{i+1}, \dots, u_i^p\}$. This way we obtain an instance for DISJOINT CONNECTED SUBGRAPHS(k), where $k \leq \sum_{i=1}^{t} |Z_i| + p(p-1) \leq p + p^2 - p = p^2$, such that Theorem 4 can be applied. We can check all possible guesses, since the total number of these guesses is bounded by $|V|^{p^2}$. □

3 Some simple polynomially solvable cases

Since K_p-MINOR CONTAINMENT and K_p-CONTRACTIBILITY are the same problem, we immediately derive the following result from Proposition 1.

Proposition 6 ([5]) *The K_p-CONTRACTIBILITY problem is solvable in polynomial time for any integer $n \geq 1$.*

The above result includes $H_8 = K_5$. We can also obtain Proposition 6 by applying Corollary 5 with $Z_i = \emptyset$ for $i = 1, \dots, p$. Also for the graphs $H_1, \dots, H_7$ more general polynomial time results will be given.

For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$, we denote their *join* by $G_1 \bowtie G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$, and their *disjoint union* by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. For the disjoint union $G \cup G \cup \dots \cup G$ of k copies of the graph G, we write shortly kG; for $k = 0$ this yields the empty graph $(\emptyset, \emptyset)$. For non-negative integers $a_1, a_2, \dots, a_k$, we let $H_1^*(a_1, a_2, \dots, a_k)$ be the graph $K_1 \bowtie (a_1 K_1 \cup a_2 P_2 \cup \dots \cup a_k P_k)$, $H_2^*(a_1, a_2)$ be the graph $K_2 \bowtie (a_1 K_1 \cup a_2 K_2)$, and $H_3^*(a_1)$ be the graph $K_3 \bowtie a_1 K_1$.

Brouwer & Veldman [2] prove that the H-CONTRACTIBILITY problem is solvable in polynomial time for any $H = H_1^*(a_1)$ or $H = H_1^*(a_1, a_2)$, where a_1, a_2 are non-negative integers. The main result in this section is that the H-CONTRACTIBILITY problem is solvable in polynomial time for $H = H_1^*(a_1, a_2, \dots, a_k), H = H_2^*(a_1, a_2)$, and $H = H_3^*(a_1)$ for any fixed set of non-negative integers $a_1, a_2, \dots, a_k$. We note that $H_1 = H_1^*(2, 1), H_2 = H_1^*(0, 2), H_3 = H_1^*(1, 0, 1), H_4 = H_1^*(0, 0, 0, 1), H_5 = H_2^*(3, 0), H_6 = H_2^*(1, 1)$ and $H_7 = H_3^*(2)$.

We prove the above polynomial time results for each class as follows. First we derive a lemma stating a number of properties of a certain H-witness structure of a H-contractible graph, when H belongs to one of the above classes. Then we will proceed as we did in the proof of Corollary 5: we perform a high (but still bounded by a polynomial in $|V_H|$) number of guesses, while for each guess we apply Corollary 5, if necessary.

Lemma 3.1 *Let y be a vertex of a connected graph H such that $H[V_H\backslash\{y\}]$ contains a component that is a path $R = x_1 \dots x_k$ on $k \geq 1$ vertices, and x_i is adjacent to y in H for $i = 1, \dots, k$. Let G be an H-contractible graph with witness structure $\mathcal{W}$. Then G owns a witness structure $\mathcal{W}'$ that has the following properties:*

(i) $W'(x_i)$ *contains exactly one vertex* u_i *that is adjacent to* $W'(y)$ *for* $i = 1, \dots, k$;

(ii) u_i *is the only vertex of* $W'(x_i)$ *that is adjacent to* $W'(x_{i+1})$ *for* $i = 1, \dots, k-1$;

(iv) $W(y) \subseteq W'(y)$;

(v) $W'(z) = W(z)$ *for all* $z \in V_H\backslash\{x_1, \dots, x_k, y\}$;

(vi) $N(W'(z)) \cap (W'(y)\backslash W(y)) = \emptyset$ *for all* $z \in V_H\backslash\{x_1, \dots, x_k, y\}$.

Proof: Let G be an H-contractible graph with witness structure $\mathcal{W}$. We define $W'(z) := W(z)$ for all $z \in V_H\backslash\{x_1, \dots, x_k, y\}$. We use induction to prove the claim. First consider $W(x_1)$. Let $u \in W(x_1)$ be adjacent to $W(y)$. We apply the following procedure repeatedly:

Case 1. The subgraph $G[W(x_1)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W(x_2)$. Then we add all vertices in $W(x_1)\backslash V_L$ to $W(y)$.

Case 2. Vertex u is the only vertex of $W(x_1)$ adjacent to $W(x_2)$, and $G[W(x_1)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W(y)$. Then we add all vertices in V_L to $W(y)$.

It is easy to check that applying any of the above two cases results in a new H-witness structure of G that leaves all witness sets $W'(z)$ for $z \notin \{x_1, y\}$ unchanged, and that does not introduce any new edges between $W(y)$ and $W'(z)$ for $z \notin \{x_1, x_2, y\}$. Each case operation reduces the size of $W(x_1)$. Hence, after at most $|W(x_1)| - 1$ single operations we find the desired set $W'(x_1)$ with desired vertex u_1.

Now suppose we have found the desired witness sets $W'(x_1), \dots, W'(x_{i-1})$ with corresponding vertices $u_1, \dots, u_{i-1}$ for some $1 \leq i \leq k-1$. Let $W^*(x_i)$ and $W^*(y)$ denote the witness sets for x_i and y, respectively, that we have obtained so far. Let u be a vertex in $G[W^*(x_i)]$ that is adjacent to $W^*(y)$. We apply the following procedure repeatedly.

Case 1. $G[W^*(x_i)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W(x_{i+1})$. If V_L is adjacent to $W'(x_{i-1})$ as well, then we add all vertices in $W^*(x_i)\backslash V_L$ to $W^*(y)$. Otherwise, we add V_L to $W(x_{i+1})$.

Case 2. Vertex u is the only vertex in $W^*(x_i)$ that is adjacent to $W(x_{i+1})$, and $G[W^*(x_i)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W^*(y)$ but not to $W'(x_{i-1})$. Then we add V_L to $W^*(y)$.

Case 3. Vertex u is the only vertex in $W^*(x_i)$ that is adjacent to $W(x_{i+1})$, subgraph $G[W^*(x_i)\backslash\{u\}]$ contains two different components L, L' whose vertex sets V_L and $V_{L'}$ are adjacent to $W'(x_{i-1})$, while V_L is adjacent to $W^*(y)$ as well. Then we add V_L to $W^*(y)$.

Case 4. Vertex u is the only vertex in $W^*(x_i)$ that is adjacent to $W(x_{i+1})$, subgraph $G[W^*(x_i)\backslash\{u\}]$ contains exactly one component L that has a vertex set V_L adjacent to $W'(x_{i-1})$, and vertex set V_L is adjacent to $W^*(y)$ as well. If vertex u is not adjacent to $W'(x_{i-1})$, then we add all vertices in $W^*(x_i)\backslash V_L$ to $W(x_{i+1})$. Otherwise we add all vertices in V_L to $W^*(y)$.

It is easy to check that applying any of the above four cases results in a new H-witness structure of G that leaves all witness sets $W'(z)$ for $z \notin \{x_i, x_{i+1}, y\}$ unchanged, that does not introduce any new edges between $W^*(y)$ and $W'(z)$ for $z \notin \{x_{i-1}, x_i, x_{i+1}, y\}$, and that does not introduce any new edges between $W'(x_{i-1})\backslash\{u_{i-1}\}$ and $W^*(y)$. Each case operation reduces the size of $W^*(x_i)$.

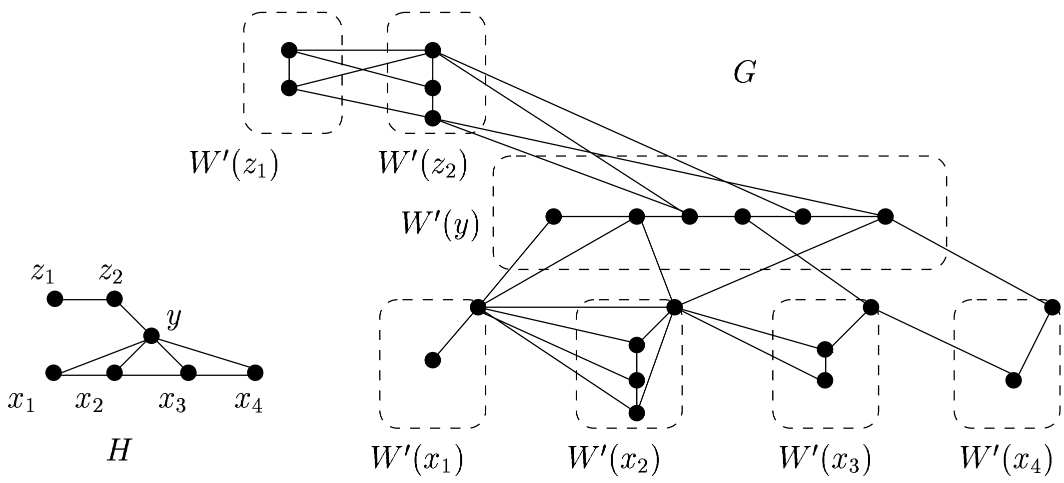

Figure 4: A graph H and an H-contractible graph G with witness structure $\mathcal{W}'$.

Hence, after at most $|W^*(x_i)| - 1$ single operations we find the desired set $W'(x_i)$ with desired vertex u_i.

Finally, let $W^*(x_k)$ denote the witness set for x_k that we have obtained from $W(x_k)$ so far. Let $u \in W^*(x_k)$ be adjacent to $W^*(y)$. We apply the following procedure repeatedly.

Case 1. The subgraph $G[W^*(x_k)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W'(x_{k-1})$. Then we add all vertices in $W^*(x_k)\backslash V_L$ to $W^*(y)$.

Case 2. Vertex u is the only vertex of $W^*(x_k)$ adjacent to $W'(x_{k-1})$, and $G[W^*(x_k)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W^*(y)$. Then we add all vertices in V_L to $W^*(y)$.

Again, it is easy to check that applying any of the above two cases results in a new H-witness structure of G that leaves all witness sets $W'(z)$ for $z \notin \{x_k, y\}$ unchanged, that does not introduce any new edges between $W^*(y)$ and $W'(z)$ for $z \notin \{x_{k-1}, x_k, y\}$ and that does not introduce any new edges between $W'(x_{k-1})\backslash\{u_{k-1}\}$ and $W^*(y)$. Each case operation reduces the size of $W^*(x_k)$. Hence, after at most $|W^*(x_k)| - 1$ single operations we find the desired set $W'(x_k)$ with desired vertex u_k. We denote the resulting witness set for y by $W'(y)$. This way we have obtained the H-witness structure $\mathcal{W}'$ of G. See Figure 4 for an example. □

We can now prove the following result.

Theorem 7 *The* $H_1^*(a_1, a_2, \ldots, a_k)$-CONTRACTIBILITY *problem is solvable in polynomial time for any fixed set of non-negative integers* $a_1, a_2, \ldots, a_k$ *for all* $k \geq 1$.

Proof: Let $G = (V, E)$ be a connected graph, and let y be the dominating vertex of $H = H^*(a_1, a_2, \ldots, a_k)$. Denote the set of components in $H[V_H\backslash\{y\}]$ by $\mathcal{P}$. Recall that all components in $\mathcal{P}$ are paths. If G is H-contractible then, by frequently applying Lemma 3.1, we find that graph G owns a witness structure $\mathcal{W}$ that satisfies the following two properties for each $P = x_1^P \ldots x_{|V_P|}^P \in \mathcal{P}$.

(i) witness set $W(x_i^P)$ contains exactly one vertex u_i^P that is adjacent to $W(y)$ for $i = 1, \ldots, |V_P|$;

(ii) vertex u_i^P is the only vertex of $W(x_i^P)$ that is adjacent to $W(x_{i+1}^p)$ for $i = 1, \ldots, k-1$.

So we only have to check whether we can find a witness structure of G satisfying (i) and (ii) for all $P \in \mathcal{P}$. We do this as follows. For each component $P \in \mathcal{P}$ we guess a set S^P of $|V_P|$ vertices

$s_1^P, \ldots, s_{|V_P|}^P$ in G, such that the *guess set* $S = \bigcup_{P \in \mathcal{P}} S^P$, i.e., the set of all guessed vertices, has cardinality $|S| = \sum_{P \in \mathcal{P}} |V_P|$.

We say that our guess set S is *appropriate* if G has a witness structure $\mathcal{W}$ that satisfies property (i) and (ii) for all $P \in \mathcal{P}$ by $s_i^P = u_i^P$ for $1 \leq i \leq |V_P|$. If it turns out that S is not appropriate, we have to determine another guess set.

In order to check whether S is appropriate we do as follows. We do not allow any two vertices s_i^P and s_j^Q to be adjacent if P, Q are two different paths in $\mathcal{P}$. We must also have guessed the vertices of S in such a way that for all $P \in \mathcal{P}$, the components of the subgraph $G[S^P]$ are paths.

For each vertex $s_i \in S$ we guess a vertex t_i in $N(s_i) \backslash S$. We denote the set of all these *guessed neighbors* by T. Two vertices in S are allowed to have the same guessed neighbor t_i in $G[V \backslash S]$ (so $|T| \leq |S|$ holds). If $N(s_i) \backslash S = \emptyset$ for some $s_i \in S$, then guess set S is not appropriate.

We define D to be the component in $G[V \backslash S]$ that contains vertex t_1. We check whether $T \subseteq V_D$ holds. In that case we consider V_D as a candidate witness set for $W(y)$. If not, then guess set S is not appropriate.

Suppose $W(y) = V_D$ can be considered as a candidate witness set. We then consider the graph $G[V \backslash V_D]$. We denote the set of components in $G[V \backslash V_D]$ by $\mathcal{K}$. If there exists a component $K \in \mathcal{K}$ that contains two vertices s_i^P and s_j^Q for $P \neq Q$, then guess set S is not appropriate. If the vertices of some set S^P are distributed over more than one component in $\mathcal{K}$, then S is not appropriate.

Suppose every component in $\mathcal{K}$ contains all vertices of exactly one set S^P (so $|\mathcal{K}| = |\mathcal{P}|$). We check each component $K \in \mathcal{K}$ as follows. Let K contain the set S^P corresponding to $P \in \mathcal{P}$. Recall that, due to our choice of S, the components of the subgraph $G[S^P]$ are all paths. For each $s \in S^P$ and each $v \in V_K \backslash S^P$, we check whether there exists a path $Q(v, s)$ from v to s in $G[(V_K \backslash S^P) \cup \{s\}]$. If we find at least three of such paths $Q(v, s_h^P), Q(v, s_i^P), Q(v, s_j^P)$ for some v, then S is not appropriate.

This way we can check in polynomial time whether S is appropriate. Since the total number of different guess sets is bounded by a polynomial in $|V_H|$ the theorem has been proven. □

Lemma 3.2 *Let y_1, y_2 be two adjacent vertices of a connected graph H such that $H[V_H \backslash \{y_1, y_2\}]$ contains a component that is a path $R = x_1 \ldots x_k$ on $1 \leq k \leq 2$ vertices such that every vertex on R is adjacent to both y_1 and y_2. Let G be an H-contractible graph with witness structure W. Then G owns a witness structure $\mathcal{W}'$ that has the following properties:*

(i) $W'(x_i)$ contains exactly one vertex u_i that is adjacent to $W'(y_1) \cup W'(y_2)$ for $i = 1, \ldots, k$;

(ii) If $k = 2$, then vertex u_1 is the only vertex of $W'(x_1)$ that is adjacent to $W'(x_2)$;

(iv) $W(y_i) \subseteq W'(y_i)$ for $i = 1, 2$;

(v) $W'(z) = W(z)$ for all $z \in V_H \backslash (V_R \cup \{y_1, y_2\})$;

(vi) $N(W'(z)) \cap (W'(y_i) \backslash W(y_i)) = \emptyset$ for all $z \in V_H \backslash (V_R \cup \{y_1, y_2\})$ and $i = 1, 2$.

Proof: We first suppose $k = 2$. Let G be an H-contractible graph with witness structure $\mathcal{W}$. We define $W'(z) = W(z)$ for all $z \in V_H \backslash \{x_1, x_2, y_1, y_2\}$. We first consider $W(x_1)$. Let $u \in W(x_1)$ be adjacent to $W(y_1)$. We apply the following procedure repeatedly:

Case 1. $G[W(x_1) \backslash \{u\}]$ contains a component L whose vertex set V_L is adjacent to $W(x_2)$. If V_L is adjacent to $W(y_2)$ as well, then we add all vertices in $W(x_1) \backslash V_L$ to $W(y_1)$. Otherwise we add all vertices in V_L to $W(x_2)$.

Case 2. Vertex u is the only vertex adjacent to $W(x_2)$, and $G[W(x_i)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W(y_1) \cup W(y_2)$. We add all vertices in V_L to $W(y_2)$ if V_L is adjacent to $W(y_2)$ and otherwise we add V_L to $W(y_1)$.

It is easy to check that applying any of the above two cases results in a new H-witness structure of G that leaves all witness sets $W'(z)$ for $z \notin \{x_1, x_2, y_1, y_2\}$ unchanged, that does not introduce any new edges between $W(y_1) \cup W(y_2)$ and $W'(z)$ for $z \notin \{x_1, x_2, y_1, y_2\}$. Each case operation reduces the size of $W(x_1)$. Hence, after at most $|W(x_1)| - 1$ single operations we find the desired set $W'(x_1)$ with desired vertex u_1.

Let $W^*(x_2), W^*(y_1)$ and $W^*(y_2)$ denote the witness sets for x_2, y_1 and y_2, respectively, that we have obtained so far. Let u be a vertex in $G[W^*(x_2)]$ that is adjacent to $W^*(y_1)$. We apply the following procedure repeatedly.

Case 1. The subgraph $G[W^*(x_2)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W^*(y_2)$. If V_L is adjacent to $W'(x_1)$ as well, then we add all vertices in $W^*(x_2)\backslash V_L$ to $W^*(y_1)$. Otherwise we add all vertices in V_L to $W^*(y_2)$.

Case 2. Vertex u is the only vertex of $W^*(x_2)$ adjacent to $W^*(y_2)$, and $G[W^*(x_2)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W^*(y_1)$. If V_L is adjacent to $W'(x_1)$ as well, then we add all vertices in $W^*(x_2)\backslash V_L$ to $W^*(y_2)$. Otherwise we add all vertices in V_L to $W^*(y_1)$.

Again, it is easy to check that applying any of the above two cases results in a new H-witness structure of G that leaves all witness sets $W'(z)$ for $z \notin \{x_2, y_1, y_2\}$ unchanged, that does not introduce any new edges between $W^*(y_1) \cup W^*(y_2)$ and $W'(z)$ for $z \notin \{x_1, x_2, y_1, y_2\}$, and that does not introduce any new edges between $W'(x_1)\backslash\{u_1\}$ and $W^*(y_1) \cup W^*(y_2)$. Each case operation reduces the size of $W^*(x_2)$. Hence, after at most $|W^*(x_2)| - 1$ single operations we find the desired set $W'(x_2)$ with desired vertex u_2. We denote the resulting witness sets for y_1 and y_2 by $W'(y_1)$ and $W'(y_2)$, respectively. This way we have obtained the H-witness structure $\mathcal{W}'$ of G.

The case $k = 1$ can be proven by using similar (but simpler) arguments. □

We are now ready to prove the following theorem.

Theorem 8 *The* $H_2^*(a_1, a_2)$-CONTRACTIBILITY *problem is solvable in polynomial time for any fixed pair of non-negative integers* a_1, a_2.

Proof: Let $G = (V, E)$ be a connected graph. Let y_1, y_2 be the two dominating vertices of $H = H_2^*(a_1, a_2)$. Denote the set of components in $H[V_H\backslash\{y_1, y_2\}]$ by $\mathcal{P}$. Recall that any component in $\mathcal{P}$ is either a vertex or an edge. If G is H-contractible then, by frequently applying Lemma 3.2, we find that graph G owns a witness structure $\mathcal{W}$ that satisfies the following two properties for each $P = x_1^P \dots x_{|V_P|}^P \in \mathcal{P}$.

(i) witness set $W(x_i^P)$ contains exactly one vertex u_i^P that is adjacent to $W(y_1) \cup W(y_2)$ for $i = 1, \dots, |V_P|$;

(ii) if $|V_P| = 2$, then vertex u_1^P is the only vertex of $W(x_1^P)$ that is adjacent to $W(x_2^p)$.

So we only have to check whether we can find a witness structure of G satisfying (i) and (ii) for all $P \in \mathcal{P}$. We do this as follows. For each component $P \in \mathcal{P}$ we guess a set S^P of $|V_P| \leq 2$ vertices $s_1^P, \dots, s_{|V_P|}^P$ in G, such that the *guess set* $S = \bigcup_{P\in\mathcal{P}} S^P$, i.e., the set of all guessed vertices, has cardinality $|S| = \sum_{P\in\mathcal{P}} |V_P|$.

We say that guess set S is *appropriate* if G has a witness structure $\mathcal{W}$ that satisfies property (i) and (ii) for all $P \in \mathcal{P}$ by $s_i^P = u_i^P$ for $1 \leq i \leq |V_P|$. If it turns out that S is not appropriate, we have to determine another guess set.

In order to check whether S is appropriate we do as follows. We do not allow any two vertices s_i^P and s_j^Q to be adjacent if P, Q are two different paths in $\mathcal{P}$.

For each vertex $s_i \in S$ we guess two different vertices t_i^1, t_i^2 in $N(s_i)\backslash S$. We denote the set of all guessed neighbors t_i^1 by T_1 and the set of all guessed neighbors t_i^2 by T_2. Two vertices in S are allowed to have some common guessed neighbor in T_1 or in T_2. If $|N(s)\backslash S| \leq 1$ for some $s \in S$, then guess set S is not appropriate.

We define D to be the component in $G[V\backslash S]$ that contains vertex t_1^1. We check whether $T_1 \cup T_2 \subseteq D$ holds. If not, then guess set S is not appropriate. Otherwise, due to Corollary 5, we can check in polynomial time whether D is K_2-contractible such that T_1 and T_2 are contained in separate witness sets. If D is not K_2-contractible in this way, then we need to guess different sets T_1 and T_2 and check again. Since the total number of different pairs T_1, T_2 is bounded by a polynomial in $|V_H|$, we find in polynomial time either that candidate witness sets $W(y_1)$ with $T_1 \subseteq W(y_1)$ and $W(y_2)$ with $T_2 \subseteq W(y_2)$ exist, or else that S is not appropriate.

Suppose candidate witness sets $W(y_1)$ and $W(y_2)$ as above exist. We now consider the graph $G[V\backslash V_D]$, and use the same procedure as in the proof of Theorem 7 to finish the verification whether S is appropriate or not.

This way we can check in polynomial time whether S is appropriate. Since the total number of different guess sets is bounded by a polynomial in $|V_H|$ the theorem has been proven. □

We finish this section by considering the $H_3^*(a_1)$-CONTRACTIBILITY problem for any fixed integer $a_1 \geq 1$. Since any $H_3^*(a_1)$ is 2-connected, we can use the following lemma, which is easy to verify.

Lemma 3.3 ([2]) *A graph G is contractible to a 2-connected graph H if and only if G is connected and some block of G is contractible to H.*

Due to this lemma it is sufficient to verify each block of a graph $G = (V, E)$. Note that we can find all blocks of a graph G in polynomial time.

Corollary 9 *Let a_1 be a non-negative integer. If the $H_3^*(a_1)$-*CONTRACTIBILITY *problem is solvable in polynomial time for the class of 2-connected graphs, then the $H_3^*(a_1)$-*CONTRACTIBILITY *problem is solvable in polynomial time.*

From now on we assume that our instance graphs are 2-connected. The following lemma simplifies the witness structure of an $H_3^*(a_1)$-contractible graph. Note that, in this lemma, $[y_1, y_3]$ does not have to be an edge in H.

Lemma 3.4 *Let y_1, y_2, y_3 be three vertices of a connected graph H with $[y_1, y_2], [y_2, y_3] \in E_H$, and let x be a vertex in H with $N(x) = \{y_1, y_2, y_3\}$. Let G be a 2-connected graph that is H-contractible with witness structure $\mathcal{W}$. Then G owns a witness structure $\mathcal{W}'$ that has the following properties:*

(i) $W'(x)$ *consists of exactly one vertex* v*;*

(ii) $W(y_i) \subseteq W'(y_i)$ *for* $i = 1, 2, 3$*;*

(v) $W'(z) = W(z)$ *for all* $z \in V_H\backslash\{x, y_1, y_2, y_3\}$*;*

(vi) $N(W'(z)) \cap (W'(y_i)\backslash W(y_i)) = \emptyset$ *for all* $z \in V_H\backslash\{x, y_1, y_2, y_3\}$ *and* $i = 1, 2, 3$.

Proof: Let G be a 2-connected, H-contractible graph with witness structure $\mathcal{W}$. Let $u \in W(x)$ be adjacent to $W(y_2)$. We apply the following procedure repeatedly:

Case 1. $G[W(x)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W(y_1)$. If V_L is adjacent to $W(y_3)$ as well, then we add all vertices in $W(x)\backslash V_L$ to $W(y_2)$. Otherwise we add all vertices in V_L to $W(y_1)$.

Case 2. Vertex u is the only vertex adjacent to $W(y_1)$, and $G[W(x)\backslash\{u\}]$ contains a component L whose vertex set V_L is adjacent to $W(y_2) \cup W(y_3)$. We add all vertices in V_L to $W(y_3)$ if V_L is adjacent to $W(y_3)$ and otherwise we add V_L to $W(y_2)$.

Applying any of the above two cases does not introduce any new edges between $W(y_1)$ and $W(y_3)$. Furthermore, it is easy to check that applying any of the above two cases results in a new H-witness structure of G that leaves all witness sets $W(z)$ for $z \notin \{x, y_1, y_2, y_3\}$ unchanged, and that does not introduce any new edges between $W(y_1) \cup W(y_2) \cup W(y_3)$ and $W(z)$ for $z \notin \{x, y_1, y_2, y_3\}$. Each case operation reduces the size of $W(x)$. Hence, after at most $|W(x)| - 1$ single operations we find a witness set $W'(x)$ with only one vertex u that is adjacent to $W(y_1) \cup W(y_2) \cup W(y_3)$. Since G is 2-connected, we conclude that $W'(x) = \{u\}$. □

We are now ready to prove the following theorem.

Theorem 10 *The* $H_3^*(a_1)$-CONTRACTIBILITY *problem is solvable in polynomial time for any fixed non-negative integer* a_1.

Proof: Let $G = (V, E)$ be a connected graph. Due to Corollary 9 we may assume that G is 2-connected. Let $Y = \{y_1, y_2, y_3\}$ be the the three dominating vertices of $H = H_3^*(a_1)$. Recall that all components in $H[V_H\backslash Y]$ are vertices.

If G is H-contractible then, by frequently applying Lemma 3.4, we find that graph G owns a witness structure $\mathcal{W}$ with $|W(x)| = 1$ for all $x \in V_H\backslash Y$. So we only have to check whether we can find such an H-witness structure of G.

We do this as follows. We guess a *guess set* S consisting of $|V_H| - 3$ vertices of G such that no two vertices in S are adjacent. We say that guess set S is *appropriate* if G has a witness structure $\mathcal{W}$ such that every witness set $W(x)$ for $x \in V_H\backslash Y$ consists of exactly one vertex of S. If it turns out that S is not appropriate, we have to determine another guess set.

In order to check whether S is appropriate we do as follows. For each vertex $s_i \in S$ we guess three different vertices t_i^1, t_i^2, t_i^3 in $N(s_i)\backslash S$. For $j = 1, 2, 3$, we denote the set of all guessed neighbors t_i^j by T_j. Two vertices in S are allowed to have some common guessed neighbor in $G[V\backslash S]$. If $|N(s)\backslash S| \leq 2$ for some $s \in S$, then guess set S is not appropriate.

We define D to be the component in $G[V\backslash S]$ that contains vertex t_1^1. We check whether $T_1 \cup T_2 \cup T_3 \subseteq D$ holds. If not, then guess set S is not appropriate. Otherwise, due to Corollary 5, we can check in polynomial time whether D has a K_3-witness structure such that the sets T_1, T_2, T_3 are contained in three separate witness sets. If this is the case then we are done. Otherwise we need to guess different sets T_1, T_2, T_3 and check again. Since the total number of different triples T_1, T_2, T_3 is bounded by a polynomial in $|V_H|$, we find in polynomial time whether S is appropriate or not.

Since the total number of different guess sets S is bounded by a polynomial in $|V_H|$, the theorem has been proven. □

4 The NP-complete cases

We prove that the H-CONTRACTIBILITY problem is NP-complete if $H = H_i$ for $i = 11, \ldots, 15$ (cf. Figure 2). Since we can guess a partition $\mathcal{W}$ of the vertex set V of an instance graph $G = (V, E)$

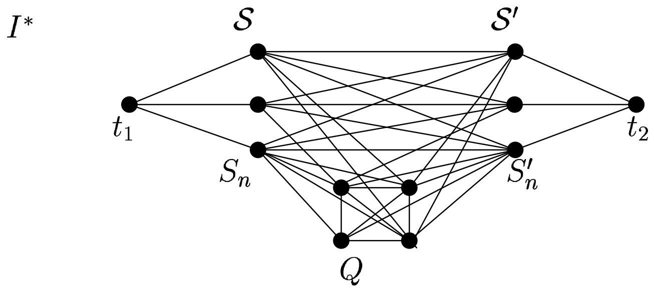

Figure 5: The P_4-identifier I^* of an instance $(Q, \mathcal{S})$.

and check in polynomial time if $\mathcal{W}$ is an H-witness structure, any H-CONTRACTIBILITY problem is a member of NP.

A *hypergraph* $(Q, \mathcal{S})$ is a set $Q = \{q_1, \ldots, q_m\}$ together with a set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of Q. A 2-*coloring* of a hypergraph $(Q, \mathcal{S})$ is a partition of Q into $Q_1 \cup Q_2$ such that $Q_1 \cap S_j \neq \emptyset$ and $Q_2 \cap S_j \neq \emptyset$ for $1 \leq j \leq n$. In our proofs we use reduction from the following, well-known NP-complete problem (cf. [3]).

HYPERGRAPH 2-COLORABILITY
Instance: A hypergraph $(Q, \mathcal{S})$.
Question: Does $(Q, \mathcal{S})$ have a 2-coloring?

With a hypergraph $(Q, \mathcal{S})$ we associate its *incidence graph* I, which is a bipartite graph on $Q \cup \mathcal{S}$, where $[q, S]$ forms an edge if and only if $q \in S$. Adding the set Q to $\mathcal{S}$ does not change the complexity of the problem. So we assume that the set Q itself is always a member of $\mathcal{S}$. Now we extend I as follows. First we insert two new vertices t_1 and t_2 and a set $\mathcal{S}'$ of n vertices $S'_1, \ldots, S'_n$, where S'_j is a copy of S_j. Then we add the following edges

- $[t_1, S]$ for all $S \in \mathcal{S}$;
- $[t_2, S']$ for all $S' \in \mathcal{S}'$;
- $[S, S']$ for all $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$;
- $[q, S'_j]$ if and only if $[q, S_j] \in E_I$;
- $[q_i, q_j]$ for all $q_i \neq q_j \in Q$.

For a reason that will be made clear in Proposition 11, we call the extended incidence graph obtained this way the P_4-*identifier* of instance $(Q, \mathcal{S})$ and denote it by I^*. See Figure 5 for an example.

The *length* of a path is the number of its edges. The *distance* $d_G(u, v)$ between two vertices u and v in a graph is the length of a shortest path between them. In our NP-completeness reductions we frequently make use of the following observation.

Observation 4.1 *Let G be an H-contractible graph with H-witness structure $\mathcal{W}$. Let r and s be vertices in G such that $r \in W(x)$ and $s \in W(y)$ for some $x, y \in V_H$. Then $d_G(r, s) \geq d_H(x, y)$.*

Proof: A shortest path from r to s in G crosses at least as many witness sets as the number of vertices of a shortest path from x to y in H. $\square$

The following result is due to [2]. It immediately implies that the P_4-CONTRACTIBILITY problem is NP-complete. For completeness we provide its proof in the appendix.

Proposition 11 ([2]) *A hypergraph $(Q, \mathcal{S})$ with $Q \in \mathcal{S}$ is 2-colorable if and only if its P_4-identifier I^* is P_4-contractible.*

In the following three theorems we prove that the H_i-CONTRACTIBILITY problem is NP-complete for $i = 11, 12, 13$.

Theorem 12 *The H_{11}-CONTRACTIBILITY problem is NP-complete.*

Proof: We write $H_{11} = (\{w_1, w_2, x, y, z\}, \{[w_1, w_2], [w_1, x], [w_2, x], [x, y], [y, z]\})$. We note that $H[\{w_1, x, y, z\}]$ is isomorphic to P_4. Let $(Q, \mathcal{S})$ be a hypergraph with $S_n = Q \in \mathcal{S}$. First we build the P_4-identifier I^* of $(Q, \mathcal{S})$. Then we construct a graph G by adding a new vertex r to I^* together with edges $[r, t_1]$ and $[r, S_n]$. We claim that $(Q, \mathcal{S})$ is 2-colorable if and only if G is H_{11}-contractible.

Suppose $(Q, \mathcal{S})$ is 2-colorable. We define the following H_{11}-witness sets of G: $W(w_1) = \{t_1\}, W(w_2) = \{r\}, W(x) = \mathcal{S} \cup Q_1, W(y) = \mathcal{S}' \cup Q_2$, and $W(z) = \{t_2\}$.

Suppose G is H_{11}-contractible with witness structure $\mathcal{W}$. Since S_n and S'_n are adjacent to all vertices in Q, we find that in G only the distances between t_1 and t_2, and between r and t_2 are at least three. The distance between t_1 and r is equal to one. Then, by Observation 4.1 we deduce that $W(z) = \{t_2\}$ and that $W(w_1)$ and $W(w_2)$ both consist of exactly one of the vertices t_1 and r. Then $G[V_G \backslash \{r\}] = I^*$ is P_4-contractible. Hence, due to Proposition 11, the instance $(Q, \mathcal{S})$ has a 2-coloring. □

Theorem 13 *The H_{12}-CONTRACTIBILITY problem is NP-complete.*

Proof: We write $H_{12} = (\{x_1, x_2, y_1, y_2, z\}, \{[x_1, y_1], [x_2, y_2], [y_1, y_2], [y_1, z], [y_2, z]\})$. We note that $H[\{x_1, x_2, y_1, y_2\}]$ is isomorphic to P_4.

Let $(Q, \mathcal{S})$ be a hypergraph with $S_n = Q \in \mathcal{S}$. First we build the P_4-identifier I^* of $(Q, \mathcal{S})$. Then we construct a graph G by adding a new vertex r and edges $[r, S_n]$ and $[r, S'_n]$. We claim that $(Q, \mathcal{S})$ is 2-colorable if and only if G is H_{12}-contractible.

Suppose $(Q, \mathcal{S})$ is 2-colorable. We define the following H_{12}-witness sets of G: $W(x_1) = \{t_1\}, W(x_2) = \{t_2\}, W(y_1) = \mathcal{S} \cup Q_1, W(y_2) = \mathcal{S}' \cup Q_2$, and $W(z) = \{r\}$.

Suppose G is H_{12}-contractible with witness structure $\mathcal{W}$. Since S_n and S'_n are adjacent to all vertices in Q, in graph G only the distance between vertices t_1 and t_2 is at least three. Then, by Observation 4.1, we assume without of generality that $W(x_1) = \{t_1\}$ and $W(x_2) = \{t_2\}$.

Since all vertices of $\mathcal{S}$ are adjacent to t_1 and all vertices of $\mathcal{S}'$ are adjacent to t_2, both witness sets $W(y_1)$ and $W(y_2)$ are not equal to $\{r\}$. Since r is only adjacent to S_n and S'_n, we then find that $G[V_G \backslash \{r\}] = I^*$ is P_4-contractible. Hence, due to Proposition 11, the instance $(Q, \mathcal{S})$ has a 2-coloring. □

Theorem 14 *The H_{13}-CONTRACTIBILITY problem is NP-complete.*

Proof: We write $H_{13} = (\{w, x_1, x_2, y, z\}, \{[w, x_1], [w, x_2], [x_1, x_2], [x_1, y], [x_2, y], [y, z]\})$. We note that $H[\{w, x_2, y, z\}]$ is isomorphic to P_4.

Let $(Q, \mathcal{S})$ be a hypergraph with $S_n = Q \in \mathcal{S}$. We may assume that $|\mathcal{S}| \geq 2$. First we build the P_4-identifier I^* of $(Q, \mathcal{S})$. Then we construct a graph G by adding a new vertex r and edges $[r, t_1], [r, S_n]$ and $[r, S'_n]$. We claim that $(Q, \mathcal{S})$ is 2-colorable if and only if G is H_{11}-contractible.

Suppose $(Q, \mathcal{S})$ is 2-colorable. We define the following H_{13}-witness sets of G: $W(w) = \{t_1\}$, $W(x_1) = \{r\}, W(x_2) = \mathcal{S} \cup Q_1, W(y) = \mathcal{S}' \cup Q_2$, and $W(z) = \{t_2\}$.

Suppose G is H_{13}-contractible with witness structure $\mathcal{W}$. Since $|\mathcal{S}| \geq 2$, graph G is 2-connected. Then, by Lemma 3.4, we may assume that $W(x_1)$ consists of a single vertex. If $W(x_1) = \{r\}$, then $G[V_G \backslash \{r\}] = I^*$ is P_4-contractible. Then, due to Proposition 11, hypergraph $(Q, \mathcal{S})$ has a 2-coloring.

Now suppose $W(x_1) \neq \{r\}$. Since S_n and S'_n are adjacent to all vertices in Q, we find that in graph G only the distance between vertex t_1 and t_2 is at least three. Then, by Observation 4.1, we know that either $W(w) = \{t_1\}$ and $W(z) = \{t_2\}$, or else that $W(w) = \{t_2\}$ and $W(z) = \{t_1\}$.

First suppose that $W(w) = \{t_1\}$ and $W(z) = \{t_2\}$. Since $W(x_1)$ consists of a single vertex not equal to r, witness set $W(x_1) = \{S\}$ for some $S \in \mathcal{S}$. By Observation 4.1 we deduce that $(\mathcal{S} \cup r) \setminus \{S\} \subseteq W(x_2)$, and that the vertices in $\mathcal{S}'$ are in $W(y)$. Then $W(x_2) = \{r\} \cup Q_1 \cup (\mathcal{S} \backslash \{S\})$ and $W(y) = Q_2 \cup \mathcal{S}'$, where Q_1 covers $\mathcal{S} \backslash \{S\}$ and Q_2 covers $\mathcal{S}'$. Since $G[\mathcal{S}]$ is edgeless and $W(x_1)$ is not adjacent to $W(x_2)$, vertex S is adjacent to a vertex in Q_1. Hence, $Q_1 \cup Q_2$ is a 2-coloring of $(Q, \mathcal{S})$. The case $W(w) = \{t_2\}$ and $W(z) = \{t_1\}$ can be proven according to similar arguments. □

If we add the edge $[t_1, t_2]$ to P_4-identifier I^* of a hypergraph $(Q, \mathcal{S})$ we obtain the *C_4-identifier* $I^\#$ of $(Q, \mathcal{S})$. We need the following result from [2] to prove NP-completeness of the H_{14}-CONTRACTIBILITY problem. It immediately implies that the C_4-CONTRACTIBILITY problem is NP-complete. For completeness we provide its proof in the appendix.

Proposition 15 ([2]) *A hypergraph $(Q, \mathcal{S})$ with $Q \in \mathcal{S}$ is C_4-contractible if and only if its C_4-identifier $I^\#$ is C_4-contractible.*

Theorem 16 *The H_{14}-CONTRACTIBILITY problem is NP-complete.*

Proof: We write $H_{14} = (\{x_1, x_2, y_1, y_2, z\}, \{[x_1, x_2], [x_1, y_1], [x_1, y_2], [x_2, y_1], [x_2, y_2], [y_1, z], [y_2, z]\})$. We note that $H[\{x_2, y_1, y_2, z\}]$ is C_4-contractible.

Let $(Q, \mathcal{S})$ be a hypergraph with $S_n = Q \in \mathcal{S}$. First, we build its C_4-identifier $I^\#$. Then we construct a graph G by adding a new vertex r together with edges $[r, t_1], [r, t_2]$ and $[r, S]$ for all $S \in \mathcal{S}$.

Suppose $(Q, \mathcal{S})$ is 2-colorable. We define the following H_{14}-witness sets of G: $W(y_1) = \{t_2\}, W(x_1) = \{r\}, W(x_2) = \{t_1\}, W(y_2) = \mathcal{S} \cup Q_1$, and $W(z) = \mathcal{S}' \cup Q_2$.

Suppose G is H_{14}-contractible with witness structure $\mathcal{W}$. We first observe that $N(t_1) = N(r)$. This implies that whenever t_1 and r are in the same witness set, we can delete r from this set without destroying the connectivity. In that case, the graph $G[V_G \backslash \{r\}]$ is H_{14}-contractible. Since H_{14} is C_4-contractible, we then find that $G[V_G \backslash \{r\}] = I^\#$ is C_4-contractible. Then we obtain a 2-coloring of $(Q, \mathcal{S})$ due to Proposition 15. From now on we assume that r and t_1 are in separate witness sets. Since G is 2-connected, we may assume that $W(x_1)$ consists of a single vertex, due to Lemma 3.4. We distinguish six cases.

Case 1. $W(x_1) = \{r\}$. Then $G[V_G \backslash \{r\}] = I^\#$ is C_4-contractible. We obtain a 2-coloring of $(Q, \mathcal{S})$ due to Proposition 15.

Case 2. $W(x_1) = \{t_1\}$. Since $N(t_1) = N(r)$, we argue in the same way as for Case 1.

Case 3. $W(x_1) = \{q\}$ for some vertex $q \in Q$. By Observation 4.1 we find that both S_n and S'_n are in $W(x_2) \cup W(y_1) \cup W(y_2)$.

Suppose $S_n \in W(x_2)$. Vertex S_n is adjacent to all the vertices in $V_G \setminus (\mathcal{S} \cup \{t_2\})$. Therefore, by Observation 4.1, witness set $W(z) \subseteq \mathcal{S} \cup \{t_2\}$. Because $\mathcal{S}$ is not adjacent to t_2, witness set $W(z)$ is either equal to $\{t_2\}$ or consists of a vertex in $\mathcal{S}$. Then, we use Observation 4.1 to find that the vertices t_1 and r must be in $W(y_1) \cup W(y_2)$. Since there are no edges between $W(y_1)$ and $W(y_2)$, the adjacent vertices t_1 and r must be in the same witness set, a contradiction.

So S_n is either in $W(y_1)$ or in $W(y_2)$. We assume without loss of generality that $S_n \in W(y_1)$. By Observation 4.1, witness set $W(y_2) \subseteq \mathcal{S} \cup \{t_2\}$. Since $\mathcal{S}$ is not adjacent to t_2, witness set $W(y_2)$ is either equal to $\{t_2\}$ or consists of a vertex in $\mathcal{S}$. In both cases, by Observation 4.1, vertices t_1, r are in $W(x_1) \cup W(x_2) \cup W(z)$. Since $W(x_1) = \{q\}$ and there are no edges between $W(x_2)$ and

$W(z)$, the adjacent vertices t_1 and r must be in the same witness set, a contradiction. Hence, we conclude that Case 3 does not occur.

Case 4. $W(x_1) = \{t_2\}$. By Observation 4.1, witness set $W(z)$ is a subset of $\mathcal{S} \cup Q$. Suppose $W(z)$ contains some vertices of $\mathcal{S}$. Then, by Observation 4.1, witness set $W(x_2)$ is a subset of $Q \cup \mathcal{S}$. Hence, there are no edges between $W(x_1)$ and $W(x_2)$, a contradiction.

So $W(z) \subseteq Q$. Then S_n and S'_n are either in $W(y_1)$ or in $W(y_2)$. Assume without loss of generality that $S_n \in W(y_1)$. Because $[S_n, S'_n]$ is an edge in G, also vertex S'_n is in $W(y_1)$. This means that $W(y_1) \cup N(W(y_1)) = V_G$, and then Observation 4.1 implies that $W(y_2)$ is empty, a contradiction. Hence, we conclude that Case 4 does not occur.

Case 5. $W(x_1) = \{S\}$ for some $S \in \mathcal{S}$. By Observation 4.1 and our assumption that the adjacent vertices r and t_1 are in separated witness sets, we may assume without loss of generality that $t_1 \in W(y_1)$ and $r \in W(x_2)$. Since $W(x_1) = \{S\}$ and $r \in W(x_2)$, we find by Observation 4.1 that $W(z) \subseteq Q$. Vertex S_n is then adjacent to vertices from all three witness sets $W(y_1)$, $W(x_2)$, and $W(z)$. Therefore, by Observation 4.1, vertex S_n is in $W(y_1)$. Since S'_n is adjacent to $S_n \in W(y_1)$ and all vertices in $W(z) \subseteq Q$, vertex S'_n is in $W(y_1)$ as well. This means that $W(y_1) \cup N(W(y_1)) = V_G$. Then Observation 4.1 implies that $W(y_2)$ is empty, a contradiction. Hence, we conclude that Case 5 does not occur.

Case 6. $W(x_1) = \{S'\}$ for some $S' \in \mathcal{S}'$. By Observation 4.1, witness set $W(z) \subseteq Q \cup \mathcal{S}' \cup \{r, t_1\}$. Because r and t_1 are not adjacent to any vertex in $Q \cup \mathcal{S}'$, we have either $W(z) \subseteq Q \cup \mathcal{S}'$ or $W(z) \subseteq \{r, t_1\}$.

First suppose $W(z) \subseteq \{r, t_1\}$. Since we assumed that r and t_1 are in separate witness sets, witness set $W(z)$ is either equal to $\{r\}$ or to $\{t_1\}$. Assume without loss of generality that $W(z) = \{r\}$. By Observation 4.1, vertices t_1 and t_2 must be in the same witness set that is adjacent to $W(z)$, say $W(y_1)$. However, this implies that $W(y_2)$ is a subset of Q due to Observation 4.1. Then $W(y_2)$ is not adjacent to $W(z)$, a contradiction.

So we obtain that $W(z) \subseteq Q \cup \mathcal{S}'$. Since the adjacent vertices r and t are in separated witness sets that must be adjacent due to Observation 4.1, we assume without loss of generality that $r \in W(y_1)$ and $t_1 \in W(x_2)$. If there is a vertex of $\mathcal{S}'$ in $W(z)$, then S_n is adjacent to all three witness sets $W(y_1)$, $W(x_2)$ and $W(z)$. Therefore, by Observation 4.1, vertex S_n is in $W(y_1)$, and consequently $W(y_1) \cup N(W(y_1)) = V_G$. Observation 4.1 implies that $W(y_2)$ is empty, a contradiction.

We are left to consider the situation in which $W(z) \subseteq Q$. Then S_n is adjacent to all three witness sets $W(y_1)$, $W(x_2)$ and $W(z)$. Therefore, S_n is in $W(y_1)$. Since $S'_n \notin W(z)$ and S'_n has neighbors in both $W(z)$ and $W(y_1)$, vertex S'_n is in $W(y_1)$ by Observation 4.1. This means that $W(y_1) \cup N(W(y_1)) = V_G$, and then Observation 4.1 implies that $W(y_2)$ is empty, a contradiction. Hence, we conclude that Case 6 does not occur. □

Up to now we have considered fourteen of the fifteen connected graphs H on five vertices that are not covered by Theorem 2. We end this section by mentioning a family of pattern graphs, for which the corresponding contractibility problem is NP-complete. The remaining graph H_{15} belongs to this family.

Let H be a connected graph. We call a graph H' with $H'[V_H] = H$ a *degree-two cover* of H if the following two conditions both hold:

1. For all $x \in V_H$:

- if x has degree one in H, then x has degree at least two in H';
- if x has degree two in H and its two neighbors in H are adjacent, then x has degree at least three in H'.

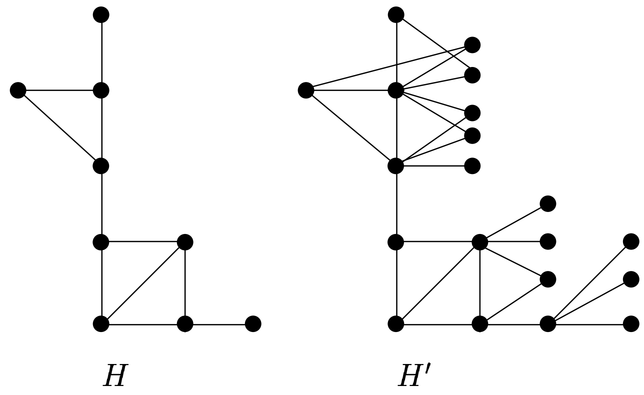

Figure 6: A graph H and one of its degree-two covers H'.

2. For all $x' \in V_{H'} \backslash V_H$:

- x' has degree one or two;
- if x' has one neighbor x, then x is in H;
- if x' has two neighbors x, y, then x and y are adjacent vertices of H.

See Figure 6 for an example. Note that a connected graph H has an infinite number of degree-two covers. Since $H'[V_H] = H$ must hold, a degree-two cover H' does not contain any new edges between vertices originally in H. This, together with condition 1, implies that every degree-two vertex in H with adjacent neighbors and every degree-one vertex in H is adjacent to at least one new vertex in H'. Due to condition 2, H' is connected too, and new vertices of H' are only adjacent to (at most two) vertices of H. Clearly, the graph H_{15} is a degree-two cover of C_4.

Theorem 17 *Let H' be a degree-two cover of a connected graph H. If the H-Contractibility problem is* NP*-complete, then the H'-Contractibility problem is* NP*-complete.*

Proof: For each $x \in V_H$, we define $M_1(x)$ to be the set of degree-one vertices in $V_{H'} \backslash V_H$ that are only adjacent to x. For each $e = [x, y] \in E_H$, we define $M_2(e)$ to be the set of degree-two vertices in $V_{H'} \backslash V_H$ that are adjacent to both x and y. Note that the union of all sets $M_1(x)$ together with all sets $M_2(e)$ form a partition of $V_{H'} \backslash V_H$. We write $p = \max\{|M_1(x)| \mid x \in V_H\}$ and $q = \max\{|M_2(e)| \mid e \in E_H\}$.

Let G be a connected graph. For each vertex $u \in V_G$, we take a set $V'(u)$ of p new vertices and add an edge between u and each vertex in $V'(u)$. For each edge $[u, v] \in E_G$, we take a set $E'([u, v])$ of q new vertices and connect each vertex in $E'([u, v])$ to both u and v by an edge. The resulting graph G' is a degree-two cover of G. We claim that G is H-contractible if and only if G' is H'-contractible.

Suppose G is H-contractible with witness structure $\mathcal{W}$. For each $x \in V_H$ we do as follows. Let u be a vertex in $W(x)$. Since $V'(u)$ contains $p \geq |M_1(x)|$ vertices, for each $x' \in M_1(x)$, we can choose a vertex $u' \in V'(u)$ in order to define a witness set $W(x') = \{u'\}$. We put any remaining vertices of $V'(u)$ in $W(x)$. For each $[x, y] \in E_H$ we do as follows. Let u be a vertex in $W(x)$ that is adjacent to a vertex $v \in W(y)$. Since $E'([u, v])$ contains $q \geq |M_2([x, y])|$ vertices, for each $x' \in M_1([x, y])$, we can choose a vertex $u' \in E'([u, v])$ in order to define a witness set $W(x') = \{u'\}$. We put any remaining vertices of $E'([u, v])$ in $W(x)$. We add all remaining vertices $v' \in V_{G'} \backslash V_G$ to the witness set of one of their corresponding neighbors in G. This way we have obtained an H'-witness structure for G'.

For the other direction of the proof, suppose G' is H'-contractible with witness structure $\mathcal{W}'$. We first show that all vertices of G' that are not in G may be removed. Let $v' \in V_{G'} \backslash V_G$. Suppose v' is in an H'-witness set of G' together with at least one other vertex. We remove v'. Then the remaining witness set is still connected and adjacent witness sets are still adjacent. This is, because v' has degree one, or v' has only two neighbors that are adjacent to each other.

Suppose v' is in an H'-witness set $W'(z)$ on its own, i.e., $W'(z) = \{v'\}$. Since v' has at most two neighbors in G', vertex z has degree at most two as well. If z has degree one, vertex z must be in $V_{H'} \backslash V_H$, by definition of H'. We remove v'. If z has two neighbors x, y in H', then v' has two neighbors u, v in G' with $u \in W'(x)$ and $v \in W'(y)$. Since u and v are adjacent by definition of G', vertices x and y are adjacent in H'. Then, by definition of H', there are only two cases to consider. Firstly, x, y are in V_H and z is in $M_2([x, y])$. We remove v'. Secondly, z is in V_H with degree one, and consequently, one of the vertices x, y, say y, is a vertex of $V_{H'} \backslash V_H$. We put all vertices of $W'(y)$ into $W'(z)$ and remove v'.

After removing all vertices of $V_{G'} \backslash V_G$, we move all non-empty witness sets $W(x')$ for $x' \in V_{H'} \backslash V_H$ to one of their adjacent witness sets. This way we have obtained an H-witness structure for G. □

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Appendix. Proofs of Propositions 11 and 15

Proof of Proposition 11: Let $S_n = Q$. We write $P_4 = z_1z_2z_3z_4$. For a 2-coloring $Q_1 \cup Q_2$ of $(Q, \mathcal{S})$, P_4-witness sets of I^* are $W(z_1) = \{t_1\}, W(z_2) = \mathcal{S} \cup Q_1, W(z_3) = \mathcal{S}' \cup Q_2$ and $W(z_4) = \{t_2\}$.

Suppose I^* is P_4-contractible with witness structure $\mathcal{W}$. Because S_n and S'_n are adjacent to all vertices in Q, the only two vertices in I^* that have distance at least three are t_1, t_2. Furthermore, $d_G(S, t_1) = 1$ for all $S \in \mathcal{S}$ and $d_G(S', t_2) = 1$ for all $S' \in \mathcal{S}'$. Then, due to Observation 4.1 we may without loss of generality assume that $W(z_1) = \{t_1\}$ and $W(z_2) = \{t_2\}$, and consequently that $W(z_2) = \mathcal{S} \cup Q_1$ and $W(z_3) = \mathcal{S} \cup Q_2$ for some partition $Q_1 \cup Q_2$ of Q. Since $W(z_2)$ and $W(z_3)$ are connected, the partition $Q_1 \cup Q_2$ is a 2-coloring of $(Q, \mathcal{S})$. □

Proof of Proposition 15: Let $S_n = Q$. We write $C_4 = z_1z_2z_3z_4z_1$. For a 2-coloring $Q_1 \cup Q_2$ of $(Q, \mathcal{S})$, C_4-witness sets of $I^{\#}$ are $W(z_1) = \{t_1\}, W(z_2) = \mathcal{S} \cup Q_1, W(z_3) = \mathcal{S}' \cup Q_2$ and $W(z_4) = \{t_2\}$.

Suppose $I^{\#}$ is C_4-contractible with witness structure $\mathcal{W}$. Suppose t_1, t_2 are in the same witness set, say $W(z_1)$. Since all vertices in $\mathcal{S} \cup \mathcal{S}'$ are adjacent to $\{t_1, t_2\}$, we find that $W(z_3) \subseteq Q$. This implies that both S_n and S'_n are in $W(z_2) \cup W(z_4)$. Since S_n and S'_n are adjacent, S_n and S'_n cannot be in two non-adjacent witness sets. Hence we assume without loss of generality that $\{S_n, S'_n\} \subseteq W(z_2)$. However, then we find that $W(z_4)$ is empty. Hence, t_1, t_2 are not in the same witness set. Since t_1, t_2 are adjacent, we then find that t_1 and t_2 are in two adjacent witness sets, say $t_1 \in W(z_1)$ and $t_2 \in W(z_2)$.

Since t_1 and S_n are adjacent, we find that $S_n \notin W(z_3)$. If $S_n \in W(z_1)$, then $W(z_3)$ cannot contain any vertices. Suppose $S_n \in W(z_2)$. Then $W(z_4) \subseteq \mathcal{S}$. Since $G[\mathcal{S}]$ is edgeless, we find that $W(z_4) = \{S\}$ for some $S \in \mathcal{S} \backslash \{S_n\}$. However, this implies that $W(z_2)$ does not contain vertices of $\mathcal{S}'$. We then find that there does not exist a path from S_n to t_2 in $W(t_2)$. Hence, $S_n \in W(z_4)$.

Since $S_n \in W(z_4)$ is adjacent to all vertices in $\mathcal{S}'$, there are no vertices of $\mathcal{S}'$ in $W(z_2)$. Since $G[W(z_2)]$ is connected, we then obtain that $W(z_2) = \{t_2\}$. By using the same arguments as above, we find that $S'_n \in W(z_3)$ and $W(z_1) = \{t_1\}$. Since t_1 is adjacent to all vertices in $\mathcal{S}$ and t_2 is adjacent to all vertices in $\mathcal{S}'$, we then obtain that $\mathcal{S} \subset W(z_4)$ and $\mathcal{S}' \subset W(z_3)$. Then $W(z_3) = Q_1 \cup \mathcal{S}'$ and $W(z_4) = Q_2 \cup \mathcal{S}$ for some partition $Q_1 \cup Q_2$ of Q. Since $G[W(z_3)]$ and $G[W(z_4)]$ are connected, partition $Q_1 \cup Q_2$ is a 2-coloring of $(Q, \mathcal{S})$. □