The computational complexity of graph contractions II: two tough polynomially solvable cases

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Abstract

For a fixed pattern graph H, let H-CONTRACTIBILITY denote the problem of deciding whether a given input graph is contractible to H. This paper is part II of our study on the computational complexity of the H-CONTRACTIBILITY problem. In the first paper we pin-pointed the complexity for all pattern graphs with five vertices except for two pattern graphs H. Here, we present polynomial time algorithms for these two remaining pattern graphs. Interestingly, in all connected cases that are known to be polynomially solvable, the pattern graph H has a dominating vertex, whereas in all cases that are known to be NP-complete, the pattern graph H does not have a dominating vertex.

Keywords: graph, edge contraction, dominating vertex, wheel, computational complexity.

2000 Mathematics Subject Classification: 05C85, 03D15.

1 Introduction

All graphs in this paper are undirected, finite, and *simple*, i.e., without loops and multiple edges. If no confusion is possible, we write $V = V_G$ and $E = E_G$ for a graph $G = (V_G, E_G)$.

Let $G = (V, E)$ be a graph, and let $e = [u, v] \in E$ be an arbitrary edge with *end-vertices* u and v. (We also say that end-vertices u and v are *adjacent*.) The *edge contraction* of edge e in G removes the two end-vertices u and v from G, and replaces them by a new vertex that is adjacent to precisely those vertices to which u or v were adjacent. We denote the resulting graph by $G \backslash e$. A graph G is contractible to a graph H (graph G is H-contractible), if the graph H can be obtained from G by a sequence of edge contractions. This leads to the following decision problem.

H-CONTRACTIBILITY
Instance: A graph $G = (V, E)$.
Question: Is G contractible to H?

A graph G is called *connected* if for every pair of distinct vertices u and v, there exists a *path* connecting u and v, i.e., a sequence of distinct vertices starting by u and ending by v where each

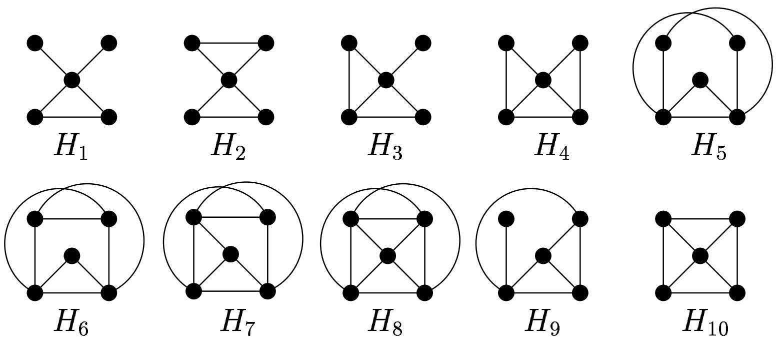

Figure 1: The graphs $H_1, H_2, \ldots, H_{10}$.

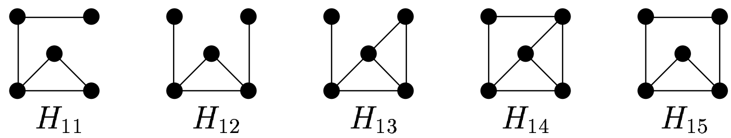

Figure 2: The graphs $H_{11}, H_{12}, \ldots, H_{15}$.

pair of consecutive vertices forms an edge of G. A *cycle* C on n vertices is a graph whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_C = \{[v_1, v_2], \ldots, [v_{n-1}, v_n], [v_n, v_1]\}$. A graph C_n denotes a cycle on n vertices. A graph that does not contain a C_3 as a subgraph is said to be *triangle-free*. A *dominating* vertex is a vertex that is adjacent to all other vertices. In [2] Brouwer & Veldman determine the complexity of the H-Contractibility problem for the following pattern graphs H.

Theorem 1 ([2]) *Let H be a connected triangle-free graph or a connected graph on at most four vertices. If H has a dominating vertex, then the H-*Contractibility *problem is polynomially solvable. If H does not have a dominating vertex, then the H-*Contractibility *problem is* NP-*complete.*

There are fifteen connected graphs H on *five* vertices that are not covered by Theorem 1; these are exactly the connected graphs on five vertices that do contain a triangle; see Figures 1 and 2 for pictures of all these graphs. In [3] the authors claimed the following result.

Theorem 2 ([3]) *Let H be a connected graph on at most five vertices. If H has a dominating vertex, then the H-*Contractibility *problem is polynomially solvable. If H does not have a dominating vertex, then the H-*Contractibility *problem is* NP-*complete.*

Due to the length of the proofs for the polynomial time algorithms for the two five-vertex graphs H_9 and H_{10}, as shown in Figure 1, we did not include these proofs in [3]. In order to prove the complete theorem we show the correctness of these polynomial time algorithms in this paper. In Section 3 we present the algorithm for the H_9-Contractibility problem, and in Section 4 we present the algorithm for the H_{10}-Contractibility problem.

2 Preliminaries

For graph terminology not defined below (or in the introduction) we refer to [1]. A graph G is a *subgraph* of a graph H, denoted by $G \subseteq H$, if $V_G \subseteq V_H$ and $E_G \subseteq E_H$. For a subset $U \subseteq V_G$ we denote by $G[U]$ the *induced subgraph* of G over U; hence $G[U] = (U, E_G \cap (U \times U))$.

Each maximal connected subgraph of a graph G is called a *component* of G. A graph $G = (V, E)$ is called *k-connected* if $G[V \backslash U]$ is connected for any set $U \subseteq V$ of at most $k-1$ vertices. A graph G that is not connected is called *disconnected*. A *k-vertex cut* is a subset $S \subseteq V$ of size k such that $G[V \backslash S]$ is disconnected. The vertex in a 1-vertex cut of a graph G is called a *cutvertex*. A vertex of a graph G that is not a cutvertex of G is called a *non-cutvertex* of G. Each maximal 2-connected subgraph of a graph G is called a *block* of G. Note that by their maximality any two blocks of G have at most one vertex (which is a cutvertex of G) in common.

For a vertex u in a graph $G = (V, E)$ we denote its *neighborhood*, i.e., the set of adjacent vertices, by $N(u) = \{v \mid [u, v] \in E\}$. The *degree* of a vertex u in G is the number of edges incident with it, or equivalently the size of its neighborhood. The *neighborhood* $N(U)$ of a subset $U \subseteq V$ is defined as $\bigcup_{u \in U} N(u) \backslash U$, and we call the vertices in $N(U)$ *neighbors* of U. If $v \in N(U)$ for some subset $U \subseteq V$ we say that v is *adjacent* to U. Two subsets $U, U' \subset V$ with $U \cap U' = \emptyset$ are *adjacent*, if there exist vertices $u \in U$ and $u' \in U'$ with $[u, u'] \in E$.

A *complete* graph is a graph with an edge between every pair of vertices. The complete graph on n vertices is denoted by K_n. A *tree* T is a connected graph that does not contain any cycles. A vertex u of degree one in a tree T is called a *leaf* of T.

If not stated otherwise, a graph P_n denotes a path on n vertices. Let $P = v_1 v_2 \dots v_p$ be a path from v_1 to v_p. Then v_{i-1} is the *precessor* of v_i on P and v_{i+1} is the *successor* of v_i on P. The vertices $v_2, \dots, v_{p-1}$ are called the *inner vertices* of P. The segment $v_i v_{i+1} \dots v_j$ is also denoted by $v_i \overrightarrow{P} v_j$. The converse segment $v_j v_{j-1} \dots v_i$ is denoted by $v_j \overleftarrow{P} v_i$. Let $C = v_0 v_1 \dots v_p v_0$ be a cycle with a fixed orientation. Then v_{i-1} is the *precessor* of v_i on C and v_{i+1} is the *successor* of v_i on C. The segment $v_i v_{i+1} \dots v_j$ is denoted by $v_i \overrightarrow{C} v_j$ where the subscripts are always to be taken modulo $|C|$. The converse segment $v_j v_{j-1} \dots v_i$ is denoted by $v_j \overleftarrow{C} v_i$. If a graph G contains an induced cycle C, i.e., $G[V_C] = C$, then we say that C is a *chordless cycle* of G.

Consider a graph $G = (V_G, E_G)$ that is contractible to a graph $H = (V_H, E_H)$. An equivalent (and for our purposes more convenient) way of stating this fact is that

- for every vertex h in V_H, there is a corresponding subset $W(h) \subseteq V$ of vertices in G such that $G[W(h)]$ is connected, and $\mathcal{W} = \{W(h) \mid h \in V_H\}$ is a partition of V_G; we call a set $W(h)$ an *H-witness set* of G for h and $\mathcal{W}$ an *H-witness structure* of G;

- for every edge $e = [h_1, h_2] \in E_H$, there is at least one edge in G that connects the vertex set $W(h_1)$ to the vertex set $W(h_2)$;

- for every two vertices h_1, h_2 in H that are not connected by an edge in E_H, there are no edges between $W(h_1)$ and $W(h_2)$.

If for every $h \in V_H$, we contract the vertices in $W(h)$ to a single vertex, then we end up with the graph H. Note that in general, these witness sets $W(h)$ are not uniquely defined (since there may be many different sequences of contractions that lead from G to H). See Figure 3 for an example. The following lemma is useful (and easy to verify).

Lemma 2.1 ([2]) *A graph G is contractible to a 2-connected graph H if and only if G is connected and some block of G is contractible to H.*

3 A tough polynomially solvable case

In this section we provide a polynomial time algorithm for the pattern graph H_9 (cf. Figure 1). Given an input graph G our algorithm either concludes that G is not H_9-contractible, or finds an

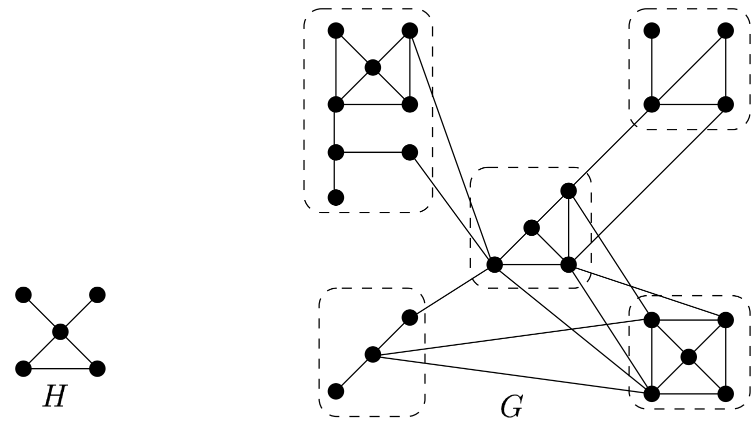

Figure 3: A graph H and an H-contractible graph G.

H_9-witness structure of G.

In H_9 we denote the vertex of degree one by x, its single neighbor by y and the other vertices by z_1, z_2, z_3. If a graph $G = (V, E)$ is H_9-contractible with witness sets $W(x), W(y_1)$ and $W(z_1), W(z_2), W(z_3)$, then we denote $X := W(x)$, $Y := W(y)$, and $Z_i := W(z_i)$ for $i = 1, 2, 3$.

3.1 Outline of the algorithm

We will act as follows.

Step 1. Increase the connectivity as much as possible

We first try to restrict ourselves to p-connected input graphs $G = (V, E)$ with p as high as possible. The intuitive reason behind this is that H_9-contractible graphs with a higher connectivity are expected to have easier to analyze witness structures than those with lower connectivity. So, if we can efficiently tear loosely connected input graphs apart into components with a higher connectivity, we might speed up the algorithm. We did not succeed in showing $p \geq 4$ but we could show that we may choose $p = 3$.

First we show that $p \geq 2$ in Lemma 3.1: if our input graph G is 1-connected then our algorithm only has to check whether G is K_4-contractible (which can be done in polynomial time). So from now on we assume that G is 2-connected. In Lemma 3.2 we then give a set of five conditions that together are necessary and sufficient for G to be H_9-contractible. In Corollary 4 we research each of these five conditions as follows. For three conditions it is immediately clear that they can be checked in polynomial time, due to a result from [3] on contracting a graph to a complete graph, where each witness set must contain certain prespecified vertices. Our algorithm checks each of these three conditions. If one of the conditions is true, then we have a 'yes'-answer. With a bit more effort we can prove that one of the two remaining conditions can also be checked in polynomial time. Our algorithm also checks this condition. If it is satisfied, then again we have a 'yes'-answer. Otherwise we are left with exactly one condition. We show that in that case our algorithm may break G into a polynomial number of smaller 3-connected parts that can be processed one by one.

Step 2. Decrease the search space of possible H_9-witness structures

As explained in Step 1, we now may assume that G is 3-connected. This has the following advantage: if G contains a chordless cycle C on at least three vertices and a vertex $v \notin V_C$ such that all neighbors of v in G do not only belong to $V \backslash V_C$ but even belong to the *same* component in $G[V \backslash (V_C \cup \{v\})]$, then G is H_9-contractible. This can be seen as follows. Let K be the component of $G[V \backslash (V_C \cup \{v\})]$ that contains $N(v)$. Since G is 3-connected, there exist at least three vertices u_1, u_2, u_3 on C that have a neighbor in K. This means we can define H_9-witness sets $X = \{v\}$, $Y = V_K$ and Z_1, Z_2, Z_3 such that $u_i \in Z_i$ for $i = 1, 2, 3$ and $Z_1 \cup Z_2 \cup Z_3 = V \backslash (V_K \cup \{v\})$. This is why we call (C, v) an H_9*-contraction pair* of G (see Figure 4 for an illustration of such contraction pair). In Lemma 3.3 we show that having a H_9-contraction pair is not only a sufficient but also a necessary condition for a 3-connected graph to be H_9-contractible. This structural result restricts the search space of possible witness structures.

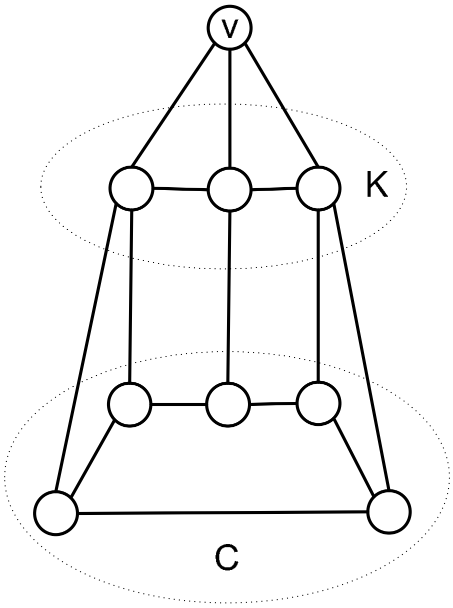

Figure 4: An example graph G with an H_9-contraction pair (C, v).

Step 3. Exclude small H_9-contraction pairs

As explained in the previous step, our algorithm now searches for a H_9-contraction pair of G. In Lemma 3.4, our algorithm uses brute force to research whether G contains a "small" H_9-contraction pair, i.e., an H_9-contraction pair (C, v) with $|V_C| \leq p$ for some fixed integer p. We set $p := 14$ (but any other value of $p \geq 14$ would be fine as well). If our algorithm finds an H_9-contraction pair (C, v) with $|V_C| \leq 14$, then we are done. If G does not have such an H_9-contraction pair, then the algorithm uses this information in the next step, as we will explain below.

Step 4. Relax the definition of an H_9-contraction pair

In this stage, we only know that G does not have a small H_9-contraction pair. In order to proceed we relax the condition that $N(v)$ must be in the same component of $G[V\backslash(V_C \cup \{v\})]$ for an H_9-contraction pair (C, v). By allowing the neighbors of v to be in *different* components of $G[V\backslash(V_C \cup \{v\})]$ we hope to find a good start pair (C, v). Our algorithm then tries to transform this start pair into an H_9-contraction pair. We call a pair (C, v), where C is a chordless cycle on at least three vertices and $v \notin V_C$ is a vertex with $N(v) \cap V_C = \emptyset$, a *pseudo-pair* of G (see Figure 5 for an illustration). In Lemma 3.5 we show that our algorithm can find a *sufficiently large* pseudo-pair, i.e., with a cycle on at least 15 vertices, of G in polynomial time, or else concludes that G does not have an H_9-contraction pair and hence is not H_9-contractible.

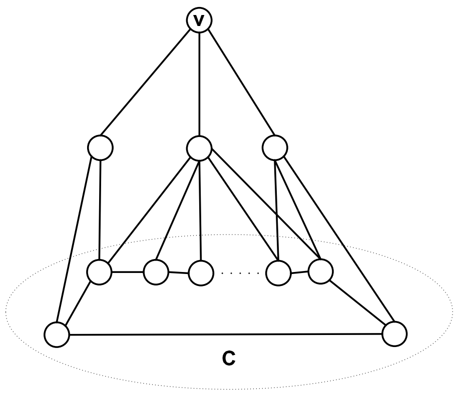

Figure 5: An example graph G with a pseudo-pair pair (C, v).

We would like to mention that the length of the cycle in a pseudo-pair of G must be sufficiently large (and a length of 15 turns out to be large enough) due to the nature of a number of operations that our algorithm performs on the pseudo-pair (in an attempt to change it into an H_9-contraction pair). We will explain these operations in the next step.

Step 5. Transform a pseudo-pair into an H_9-contraction pair

In this stage, G has a pseudo-pair (C, v) with $|V_C| \geq 15$. Our algorithm starts to perform a number of checks that each can be performed in polynomial time. Our goal is to either find an H_9-contraction pair of G, or else to find a well-structured pseudo-pair of G, namely a pseudo-pair (C', v') that satisfies the following two conditions:

(i) every vertex in $V\backslash(V_{C'} \cup \{v'\})$ is adjacent to v', so $N(v') = V\backslash(V_{C'} \cup \{v'\})$;

(ii) every vertex in $V\backslash(V_{C'} \cup \{v'\})$ is adjacent to all vertices on C'.

We call a pseudo-pair that satisfies conditions (i) and (ii) above a *complete* pseudo-pair of G (see Figure 6). The advantage of finding a complete pseudo-pair will be made clear in the next step. We first briefly explain the checks we need to perform on pseudo-pair (C, v). If (C, v) passes a check then our algorithm finds an H_9-contraction pair. Each time (C, v) fails a check, we find that (C, v) is more similar to a complete pseudo-pair, and we proceed with the next check. If (C, v) fails the last check, then (C, v) must be a complete pseudo-pair. There are five checks.

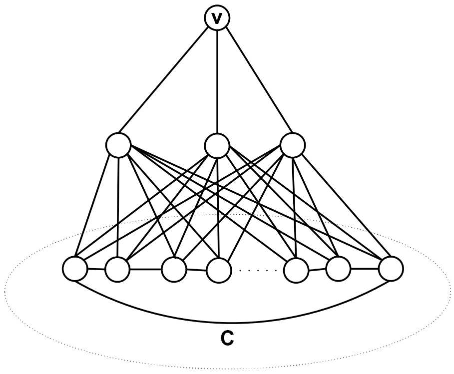

Figure 6: An example graph G with a complete pseudo-pair (C, v) that is not an H_9-contraction pair.

Check 1. In Lemma 3.6 we show how our algorithm finds an H_9-contraction pair of G if $G[V \backslash V_C]$ is not connected.

Check 2. Assume that $G[V \backslash V_C]$ is connected. Denote the set of components in $G[V \backslash (V_C \cup \{v\})]$ by $\mathcal{K}$. Lemma 3.7 shows how our algorithm finds an H_9-contraction pair of G if C contains a vertex not adjacent to the vertex set of some component $K \in \mathcal{K}$.

Check 3. Assume that all vertices on C have a neighbor in every component of $\mathcal{K}$. Using Lemma 3.8, Lemma 3.9 shows how our algorithm finds an H_9-contraction pair of G if there exists a component $K \in \mathcal{K}$ with no vertex adjacent to at least two vertices of C.

Check 4 Assume that all components in $\mathcal{K}$ contain a vertex that has at least two neighbors on C. Using Lemma 3.10, Lemma 3.11 shows how our algorithm finds an H_9-contraction pair of G if there exists a component $K \in \mathcal{K}$ that contains a vertex adjacent to at least two vertices on C but not to all of them.

Check 5 Assume that all component in $\mathcal{K}$ contain a vertex that is adjacent to all vertices on C. Lemma 3.12 shows how our algorithm finds an H_9-contraction pair of G if there exists a component in $\mathcal{K}$ that contains a vertex that is not adjacent to all vertices on C or that is not adjacent to v.

If (C, v) fails Check 5 then (C, v) must be a complete pseudo-pair of (C, v).

Step 6. Transform a complete pseudo-pair into an H_9-contraction pair

As we explained in the previous step, we now have a complete pseudo-pair (C, v) of G. In Lemma 3.13 we show that G has an H_9-contraction cycle if and only if $G[N(v)]$ contains a cycle on at most $|N(v)| - 2$ vertices. This final condition is easy to check, and in Theorem 5 we prove the correctness of the whole algorithm and show that it runs in polynomial time.

3.2 The algorithm itself

As we explained in the previous section, our polynomial time algorithm for solving the H_9-Contractibility problem performs six steps. Here we describe them in more detail.

Step 1. Increase the connectivity as much as possible

We start with the following lemma. It shows that we may restrict ourselves to 2-connected input graphs.

Lemma 3.1 *Let u be a cutvertex of a connected graph $G = (V, E)$. Then G is H_9-contractible if and only if G is K_4-contractible.*

Proof: Let $\mathcal{C}$ denote the set of all components in $G[V \backslash \{u\}]$. Suppose G is contractible to H_9. Obviously, G is contractible to K_4.

Suppose G is contractible to K_4. Since u is a cutvertex of G, the vertices of the K_4-witness sets of G not containing u all belong to one component F in $\mathcal{C}$. Let $F' \neq F$ be another component in $\mathcal{C}$. Let A be the K_4-witness set of G that contains u. Then F' is a subset of A. We choose $X = V_{F'}$ and $Y = A \backslash V_{F'}$. All other witness sets stay the same. This way we have obtained H_9-witness sets for G. □

To further increase the connectivity of our input graphs we need amongst others the following lemma. In this lemma we use the following notations. For an induced subgraph F of a graph $G = (V, E)$ and two vertices $u, v \in V \backslash V_F$, we denote $G[V_F \cup \{u, v\}]$ by $F + \{u, v\}$ and we write $F + [u, v]$ to denote the graph obtained from $F + \{u, v\}$ after adding the edge $[u, v]$. Note that $F + [u, v] = F + \{u, v\}$ if and only if $[u, v]$ is an edge in G. The graph obtained from $F + [u, v]$ after contracting $[u, v]$ is denoted by $F + uv$.

Lemma 3.2 *Let $\{u, v\}$ be a 2-vertex cut of a 2-connected graph $G = (V, E)$. Let $\mathcal{C}$ be the set of components of $G[V \backslash \{u, v\}]$. Then G is H_9-contractible if and only if at least one of the following cases is true:*

- *(i) there exists a component $F \in \mathcal{C}$ such that the $F + [u, v]$ is H_9-contractible, or*
- *(ii) there exists a component $F \in \mathcal{C}$ such that $F + \{u, v\}$ is K_4-contractible with u, v in the same K_4-witness set, or*
- *(iii) there exist components $F, F' \in \mathcal{C}$ ($F \neq F'$) such that $F + uv$ is K_4-contractible, and $F' + \{u, v\}$ is P_2-contractible with u, v in the same P_2-witness set, or*
- *(iv) $|\mathcal{C}| \geq 3$ and there exists a component $F \in \mathcal{C}$ such that $F + uv$ is K_4-contractible, or*
- *(v) there exist components $F, F' \in \mathcal{C}$ ($F \neq F'$) such that $F + [u, v]$ is K_4-contractible with u, v in two different K_4-witness sets and $F' + \{u, v\}$ is P_3-contractible with u, v in two different P_3-witness sets.*

Proof: Suppose G is H_9-contractible. We show that at least one of the cases $(i) - (v)$ is true. Let u be in witness set $W(i)$ and let v be in witness set $W(j)$. We first show the following: If $i \neq j$, then i and j must be adjacent vertices in H. Suppose otherwise, i.e., that $i = x$ and $j = z_k$ for some $1 \leq k \leq 3$, say $j = z_1$. Since y, z_2, z_3 are adjacent to each other, all vertices in $W(y) \cup W(z_2) \cup W(z_3)$ belong to the same component $F_1 \in \mathcal{C}$. Suppose some of the vertices in $W(z_1) \backslash \{v\}$ belong to some component $F_2 \in \mathcal{C}$. Then v is a cutvertex of G. This is not possible, since G is 2-connected. Hence $W(z_1) \subseteq V_{F_1} \cup \{v\}$. Suppose some of the vertices in $W(x) \backslash \{u\}$ belong to some component $F_3 \in \mathcal{C}$. Then u would be a cutvertex of G. Hence $W(x) \subseteq V_{F_1} \cup \{u\}$. This implies that $V = V_{F_1} \cup \{u, v\}$. But then $\{u, v\}$ would not be a 2-vertex cut.

Due to the above, we need to distinguish three cases.

Case 1. $\{x\} \subseteq \{i,j\} \subseteq \{x,y\}$ or $\{i,j\} \subset \{z_1,z_2,z_3\}$. Since $\{u,v\}$ is a 2-vertex cut, the vertices in the other witness sets must all belong to the same component F of $\mathcal{C}$. We remove all vertices in $W(i) \cup W(j)$ not in $V_F \cup \{u,v\}$. If $i=j$, the resulting witness set $W'(i)$ might be disconnected. If $i \neq j$, there might not be an edge between the resulting witness sets $W'(i)$ and $W'(j)$. However, in both cases, after adding the edge $[u,v]$ we obtain an H_9-contractible graph. So (i) is valid.

Case 2. $i=j=y$. Since $\{u,v\}$ is a 2-vertex cut, the vertices in the witness sets Z_1, Z_2, Z_3 must all belong to the same component F of $\mathcal{C}$. What about the vertices in X?

Suppose $X \subset V_F$. Then we remove all vertices in Y not in $V_F \cup \{u,v\}$. Then the resulting set Y' might be disconnected. However, after adding the edge $[u,v]$ we obtain an H_9-contractible graph. So (i) is valid.

Suppose $X \subseteq V_{F'}$ for some component $F' \in \mathcal{C}$ with $F' \neq F$. Suppose $G[Y]$ contains a path from u to v only using vertices from $V_F \cup \{u,v\}$. Then $F + \{u,v\}$ is K_4-contractible with u,v in the same K_4-witness set. So (ii) is valid. Suppose $G[Y]$ contains a path from u to v only using vertices from $V_{F'} \cup \{u,v\}$. Then $F' + \{u,v\}$ is P_2-contractible with u,v in the same P_2-witness set where X is contracted to one P_2-witness set and $Y \cap (V_{F'} \cup \{u,v\})$ is contracted to the other P_2-witness set. Furthermore, $F + uv$ is K_4-contractible. So (iii) is valid. In the remaining case $G[Y]$ contains a path from u to v only using vertices from $V_{F''} \cup \{u,v\}$ for some component $F'' \in \mathcal{C} \backslash \{F, F'\}$. Then $F + uv$ is K_4-contractible. So (iv) is valid.

Case 3. $\{i,j\} = \{y, z_h\}$ for some $1 \leq h \leq 3$. We assume without loss of generality that $i=y$ and $j=z_1$. Since $\{u,v\}$ is a 2-vertex cut, the vertices in the witness sets Z_2, Z_3 must all belong to the same component F of $\mathcal{C}$. What about the vertices in X?

Suppose $X \subset V_F$. Then we remove all vertices in $Y \cup Z_1$ not in $V_F \cup \{u,v\}$. Then there might not be an edge between the resulting witness sets Y' and Z_1'. However, after adding the edge $[u,v]$ we obtain an H_9-contractible graph. So (i) is valid.

Suppose $X \subseteq V_{F'}$ for some component $F' \in \mathcal{C}$ with $F' \neq F$. Then $G[Y \cup Z_1]$ contains a path from u to v only using vertices from $V_{F'} \cup \{u,v\}$. Otherwise $\{u\}$ would be a cutvertex of G contradicting our assumption that G is 2-connected. Hence $F' + \{u,v\}$ is P_3-contractible with u,v in two different P_3-witness sets, where X is contracted into one P_3-witness set, $Y \cap (V_{F'} \cup \{u\})$ is contracted to the middle P_3-witness set, and $Z_1 \cap (V_{F'} \cup \{v\})$ is contracted into the last P_3-witness set. Furthermore, $F + [u,v]$ is K_4-contractible with u,v in two different K_4-witness sets. So (v) is valid.

It is easy to see that G is H_9-contractible if at least one of the cases (i)-(v) hold. □

In order to prove that we may restrict ourselves to the class of 3-connected graphs we also need a result on the following decision problem.

K_p-FIXED CONTRACTIBILITY
Instance: A graph $G=(V,E)$ and subsets $Z_1, \ldots, Z_t \subseteq V$ such that $\sum_{i=1}^{t} |Z_i| \leq p$.
Question: Can G be contracted to K_p with K_p-witness sets $U_1, \ldots U_p$ such that $Z_i \subseteq U_i$ for $1 \leq i \leq t$?

Proposition 3 ([3]) *The K_p-FIXED CONTRACTIBILITY problem is solvable in polynomial time.*

We note that for every 'yes'-instance of the K_p-FIXED CONTRACTIBILITY problem appropriate K_p-witness sets can be found in polynomial time as well. Both Proposition 3 and this stronger result are a direct consequence of the polynomial time algorithm in [4] on the so-called DISJOINT CONNECTED SUBGRAPHS(k) problem. See [3, 4] for more details.

Corollary 4 *If the H_9-CONTRACTIBILITY problem is solvable in polynomial time for the class of 3-connected graphs, then the H_9-CONTRACTIBILITY problem is solvable in polynomial time.*

Proof: Let G be a connected graph. If G contains a cutvertex, then we only need to check whether G is K_4-contractible due to Lemma 3.1. By Theorem 1 this can be done in polynomial time.

So we may assume that G is 2-connected. Suppose G is not 3-connected. Then we can find a 2-vertex cut $\{u, v\}$ with set $\mathcal{C}$ of components in $G[V \backslash \{u, v\}]$ in polynomial time. We first check whether one of the cases (ii)-(v) of Lemma 3.2 is valid. In polynomial time we can either find appropriate witness sets or else conclude that cases (ii)-(v) do not hold. We can do this as follows. We can check case (ii)-(iv) in polynomial time due to Proposition 3. We can check case (v) in polynomial time as follows. For each $F \in \mathcal{C}$ we can check in polynomial time if $F + [u, v]$ is K_4-contractible with u, v in two different K_4-witness sets due to Proposition 3. If so, then we consider each component $F' \in \mathcal{C} \backslash \{F\}$. We show that $F' + \{u, v\}$ is P_3-contractible with u, v in two different P_3-witness sets if and only if one of the following conditions holds (both can be checked in polynomial time): there exists a vertex $w_1 \in V_{F'}$ that is not adjacent to u such that $G[V_{F'} \backslash \{w_1\}]$ is connected, or there exists a vertex $w_2 \in V_{F'}$ that is not adjacent to v such that $G[V_{F'} \backslash \{w_2\}]$ is connected.

Suppose $F' + \{u, v\}$ is P_3-contractible with u in P_3-witness set A, and v in P_3-witness set B with $A \neq B$. Just as in the proof of Lemma 3.2, we know that A and B are adjacent. Let $P = p_1 p_2 p_3$. Then without loss of generality we assume that $A = W(p_1)$ and $B = W(p_2)$. Since we can move vertices from $W(p_3)$ to $W(p_2)$ if necessary, we may assume that $W(p_3) = \{w\}$ for some $w \in V_{F'}$. In the same way, since u is not a cutvertex of $F + \{u, v\}$, we may assume that $W(p_1) = \{u\}$. The reverse implication is trivial.

Suppose none of the cases (ii)-(v) hold. We then check whether case (i) is valid. If a graph $F + [u, v]$ for some component $F \in \mathcal{C}$ is not 3-connected, then it is at least 2-connected. Otherwise, i.e., if $F + [u, v]$ contains a cutvertex a, then a would be a cutvertex in G. We just repeat the procedure above. Any vertex that does not belong to some 2-vertex cut in G is in exactly one 3-connected graph obtained this way. Furthermore, any such 3-connected graph contains at least one vertex that is not in any 2-vertex cut in G. Hence we then find a total set of $O(|V|)$ 3-connected graphs. Each of these graphs has at most $|V|$ vertices. We check each of them. □

Step 2. Decrease the search space of possible H_9-witness structures

Due to Corollary 4 from now on we will only consider 3-connected input graphs. Recall that an H_9-contraction pair is a pair (C, v), where C is a chordless cycle on at least three vertices and v is a vertex not on C such that all neighbors of v in G belong to the same component of $G[V \backslash (V_C \cup \{v\})]$. In the next lemma we show that having an H_9-contraction pair is a sufficient and a necessary condition for a 3-connected graph G to be H_9-contractible.

Lemma 3.3 *A 3-connected graph G is H_9-contractible if and only if G has an H_9-contraction pair (C, v).*

Proof: Let $G = (V, E)$ be a 3-connected graph. First suppose G contains an H_9-contraction pair (C, v). Let K be the component of $G[V \backslash (V_C \cup \{v\})]$ that contains $N(v)$. Since G is 3-connected, there exist at least three vertices u_1, u_2, u_3 on C that have a neighbor in K. This means we can define H_9-witness sets $X = \{v\}$, $Y = V_K$ and Z_1, Z_2, Z_3 such that $u_i \in Z_i$ for $i = 1, 2, 3$ and $Z_1 \cup Z_2 \cup Z_3 = V \backslash (V_K \cup \{v\})$.

To show the other direction of the claim, suppose G is H_9-contractible with H_9-witness sets X, Y and Z_1, Z_2, Z_3. We show that $G[Z_1 \cup Z_2 \cup Z_3]$ contains a chordless cycle C with $V_C \cap Z_i \neq \emptyset$ for $i = 1, 2, 3$. Then we can take an arbitrary vertex $v \in X$, and we have an H_9-contraction pair (C, v) of G. Since Z_1, Z_2, Z_3 are H_9-witness sets corresponding to $W(z_1), W(z_2), W(z_3)$, we can determine

C as follows. Let $z_3' \in Z_3$ be adjacent to $z_1 \in Z_1$. Let P_1 be a path in $G[Z_1]$ from z_1 to the first vertex z_1' that is adjacent to a vertex $z_2 \in Z_2$ (so $z_1 = z_1'$ is possible). Let P_2 be a path in $G[Z_2]$ from z_2 to the first vertex z_2' of Z_2 that is adjacent to a vertex $z_3 \in Z_3$ (so $z_2 = z_2'$ is possible). We let P_3 be a shortest path in $G[Z_3]$ from z_3 to z_3'. Now consider $C' = z_1\overrightarrow{P_1}z_1'z_2\overrightarrow{P_2}z_2'z_3\overrightarrow{P_3}z_3'z_1$. If C' is chordless we choose $C = C'$. Suppose C' is not chordless. By construction, there are no vertices of P_1 adjacent to P_2 except z_1'. Similarly, there are no vertices from P_2 adjacent to P_3 except z_2'. By construction, all paths P_1, P_2, P_3 are induced paths in G. Hence, there exist at least one edge between P_1 and P_3 not equal to $[z_1, z_3']$. Starting in z_3 let z_3^* be the first vertex on P_3 adjacent to a vertex z_1^* on P_1. Then we choose $C = z_1^*\overrightarrow{P_1}z_1'z_2\overrightarrow{P_2}z_2'z_3\overrightarrow{P_3}z_3^*z_1^*$. □

Step 3. Exclude small H_9-contraction pairs

Due to Lemma 3.3, our goal is to construct a polynomial time algorithm that either finds an H_9-contraction pair for a 3-connected graph G, or else concludes that G does not have such a pair. The following lemma holds for any fixed integer p, but in the correctness proof of our algorithm a value of $p = 14$ turns out to be sufficiently large.

Lemma 3.4 *Let $G = (V, E)$ be a 3-connected graph. It is possible in polynomial time either to find an H_9-contraction pair of G with $|V_C| \leq 14$, or else to conclude that G does not have an H_9-contraction pair (C, v) with $|V_C| \leq 14$.*

Proof: For each subset $U \subseteq V$ of at most 14 vertices and vertex $v \notin U$ we can check in polynomial time whether $(G[U], v)$ is an H_9-contraction pair of G. The number of such pairs (U, v) is bounded by $O(|V|^{15})$. □

Step 4. Relax the definition of an H_9-contraction pair

Recall that a pseudo-pair in a 3-connected graph G is a pair (C, v), where C is a chordless cycle on at least three vertices and v is a vertex of $V \backslash V_C$ with $N(v) \cap V_C = \emptyset$. So an H_9-contraction pair (C, v) can be seen as a special kind of pseudo-pair, namely with $N(v)$ in the same component of $G[V \backslash (V_C \cup \{v\})]$. In case we do not find an H_9-contraction pair (C, v) with $|V_C| \leq 14$, we try to find a pseudo-pair (C', v') of G with $|V_{C'}| \geq 15$.

Lemma 3.5 *Let G be a 3-connected graph that does not have an H_9-contraction pair (C, v) with $|V_C| \leq 14$. Then it is possible in polynomial time either to find a pseudo-pair (C', v') of G with $|V_{C'}| \geq 15$, or else to conclude that G does not have an H_9-contraction pair.*

Proof: Let $G = (V, E)$ be a 3-connected graph that does not have an H_9-contraction pair (C, v) with $|V_C| \leq 14$. In that case we act as follows. For each set of fifteen vertices $v, u_1, \ldots, u_{14}$ of G with $u_1, \ldots, u_{14}$ inducing a path $P = u_1u_2 \ldots u_{14}$ in G and $\{u_1, \ldots, u_{14}\} \cap N(v) = \emptyset$, we remove $v, u_2, \ldots, u_{13}$, together with $N(v)$ and $N(\{u_2, \ldots, u_{13}\}) \backslash \{u_1, u_{14}\}$. Then we compute (in polynomial time) a shortest path P' from u_{14} to u_1 in the resulting graph G'. Note that $[u_{14}, u_1]$ is not an edge in G. Hence, if such an induced path P' of G' indeed exists, then $C = u_1\overrightarrow{P}u_{14}\overrightarrow{P'}u_1$ is a chordless cycle on at least fifteen vertices in $G[V \backslash (\{v\} \cup N(v))]$. Hence, (C, v) is the desired pseudo-pair. Otherwise we will guess another set of fifteen vertices of G and so on.

Obviously, G does not contain a pseudo-pair (C, v) with $|V_C| \geq 15$ if we have not found such a pair after considering all $O(|V|^{15})$ possible combinations. In that case, since any H_9-contraction pair is a pseudo-pair as well, graph G does not have an H_9-contraction pair. □

Step 5. Transform a pseudo-pair into an H_9-contraction pair

Recall that a pseudo-pair (C, v) is a complete pseudo-pair of $G = (V, E)$ if every vertex in $V \backslash (V_C \cup \{v\})$ is adjacent to all vertices in $V_C \cup \{v\}$. That is, if the following two conditions hold:

(i) every vertex in $V \backslash (V_C \cup \{v\})$ is adjacent to v, so $N(v) = V \backslash (V_C \cup \{v\})$;

(ii) every vertex in $V \backslash (V_C \cup \{v\})$ is adjacent to all vertices on C.

By a sequence of lemmas we prove that we may restrict ourselves to complete pseudo-pairs, i.e., if a pseudo-pair of a 3-connected graph G is not complete, then we can identify an H_9-contraction pair of G in polynomial time. Before we start to prove this we make the following remark.

Remark 1 Let C be a (not necessarily chordless) cycle on at least three vertices of a 3-connected graph G. Let v be a vertex in $V \backslash V_C$ such that all neighbors of v in G belong to the same component K of $G[V \backslash (V_C \cup \{v\})]$. If C is not chordless, then formally (C, v) is not an H_9-contraction pair. However, it is easy to see that we can use (C, v) to find an H_9-witness structure in polynomial time: since G is 3-connected, there exist at least three vertices u_1, u_2, u_3 on C that have a neighbor in K, and we can define H_9-witness sets $X = \{v\}$, $Y = V_K$ and Z_1, Z_2, Z_3 such that $u_i \in Z_i$ for $i = 1, 2, 3$ and $Z_1 \cup Z_2 \cup Z_3 = V \backslash (V_K \cup \{v\})$. This is why we call such a pair (C, v) an H_9-contraction pair as well. As a matter of fact, we can show that C contains a cycle C' such that (C', v) is a "real" H_9-contraction pair of G by using exactly the same arguments as in the proof of Lemma 3.3.

As we explained in Section 3.1, the first check we perform on a pseudo-pair (C, v) of a 3-connected graph G is whether $G[V \backslash V_C]$ is connected.

Lemma 3.6 *Let (C, v) be a pseudo-pair of a 3-connected graph $G = (V, E)$ with $C = u_1 u_2 \dots u_p u_1$ for some $p \geq 15$. If $G[V \backslash V_C]$ is not connected, then it is possible to find an H_9-contraction pair of G in polynomial time.*

Proof: Let $\mathcal{K}_1$ denote the set of components of $G[V \backslash (V_C \cup \{v\})]$ that contain a neighbor of v. If $|\mathcal{K}_1| = 1$, then (C, v) is an H_9-contraction pair and we are done. So assume $|\mathcal{K}_1| \geq 2$. Let $\mathcal{K}_2$ denote the set of all other components of $G[V \backslash (V_C \cup \{v\})]$, and let $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$. Suppose $G[V \backslash V_C]$ is not connected. This is equivalent to saying $|\mathcal{K}_2| \geq 1$. Below we show how we can identify an H_9-contraction pair of G. It is easy to check that this can be done in polynomial time.

First we make an observation. Since G is 3-connected, the vertex set of every component in $\mathcal{K}_1$ is adjacent to at least two different vertices on C, and the vertex set of every component in $\mathcal{K}_2$ is adjacent to at least three different vertices on C.

Let L be a component of $\mathcal{K}_2$. As we observed above, the vertex set V_L has at least three neighbors u_i, u_j, u_k on C with $i < j < k$. Let $P_1 = u_i \overrightarrow{C} u_j$, $P_2 = u_j \overrightarrow{C} u_k$, and $P_3 = u_k \overrightarrow{C} u_i$.

Suppose there exists an index $1 \leq h \leq 3$ such that every component $K \in \mathcal{K}_1$ has a neighbor in $V_C \backslash V_{P_h}$. We assume without loss of generality that $h = 1$. Let $x \in V_L$ be a neighbor of u_i, and let $y \in V_L$ be a neighbor of u_j. Let P be a path from x to y in L. Then we construct the cycle $C' = x \overrightarrow{P} y u_j \overleftarrow{C} u_i x$.

With an eye on Remark 1, we claim that (C', v) is an H_9-contraction pair of G. By definition of C', we note that $N(v) \cap V_{C'} = \emptyset$. We show that every vertex in $N(v)$ is in the same component of $G[V \backslash (V_{C'} \cup \{v\})]$. Let z, z' be two neighbors of v. Assume that $z \in V_K$ and $z' \in V_{K'}$ for some $K, K' \in \mathcal{K}_1$. Then z and z' are connected to each other via a path using only vertices from K, K' and $V_C \backslash V_{P_1}$. Hence, z and z' are in the same component of $G[V \backslash (V_{C'} \cup \{v\})]$.

If an index h as above does not exist, then $\mathcal{K}_1$ must contain three components K_1, K_2, K_3 such that $N(V_{K_i}) \cap V_C \subseteq V_{P_i}$ for $i = 1, 2, 3$. Recall that V_{K_i} is adjacent to at least two vertices of P_i for $i = 1, 2, 3$.

Case 1. There exists an index $1 \leq i \leq 3$ such that V_{K_i} is adjacent to some inner vertex of P_i.

We assume without loss of generality that $i = 1$. Then $G[V_{K_1} \cup V_{P_1}]$ contains a cycle C' that does not use both vertices u_i, u_j. We assume without loss of generality that vertex u_j is not on C'.

Let w be a non-cutvertex of K_2. We claim that (C', w) is an H_9-contraction pair of G. By definition of C' and K_2, we note that $N(w) \cap V_{C'} = \emptyset$. We show that $N(w)$ is in the component F of $G[V \backslash (V_{C'} \cup \{w\})]$ to which u_j belongs. Let t be a neighbor of w.

Clearly, P_2 is a subgraph of F. If t is on C then t is on P_2 and hence t is in F. Suppose t is in K_2. Since G is 3-connected, component K_2 contains at least two vertices w', w'' that are adjacent to $P_2 \subseteq F$. At least one of these vertices, say w', is not equal to w. Since vertex w is a non-cutvertex in K_2, there exists a path from t to w' in K_2 that does not use w. Hence, also in this case, t is in F. Suppose $t = v$. Since $K_3 \in \mathcal{K}_1$, vertex v is adjacent to a vertex t' in K_3. Recall that K_3 is adjacent to at least two vertices u_r, u_s on P_3. At least one of those vertices, say u_r, is not equal to u_i. Then it is clear that v is connected to u_j in $G[V \backslash (V_{C'} \cup \{w\})]$ via a path using only vertices of $V_{K_3} \cup (V_{P_3} \backslash \{u_i\}) \cup V_{P_2}$. This means that v is in F. Hence, we conclude that $N(w)$ is in F.

Case 2. $N(V_{K_1}) \cap V_C = \{u_i, u_j\}$ and $N(V_{K_2}) \cap V_C = \{u_j, u_k\}$ and $N(V_{K_3}) \cap V_C = \{u_k, u_i\}$ and $(N(V_L) \cap V_C) \setminus \{u_i, u_j, u_k\} \neq \emptyset$.

Suppose $u_\ell \in V_C \backslash \{u_i, u_j, u_k\}$ is adjacent to V_L. We assume without loss generality that $k < \ell$. We define $P'_1 = P_1 = u_i \overrightarrow{C} u_j, P'_2 = P_2 = u_j \overrightarrow{C} u_k, P'_3 = u_k \overrightarrow{C} u_\ell$, and $P'_4 = u_\ell \overrightarrow{C} u_i$. Then we may assume that $\mathcal{K}_1$ contains components $K'_1, \ldots K'_4$ such that $N(V_{K'_i}) \cap V_C \subseteq V_{P_i}$ for $i = 1, \ldots, 4$, Otherwise we can construct an H_9-contraction pair of G in the same way as before. Then all neighbors that $V_{K'_3}$ has on C are on $P'_3 \subset P_3$. Since there are at least two of such neighbors, $V_{K'_3}$ is adjacent to some inner vertex of P_3. This brings us back to Case 1.

Case 3. $N(V_{K_1}) \cap V_C = \{u_i, u_j\}$ and $N(V_{K_2}) \cap V_C = \{u_j, u_k\}$ and $N(V_{K_3}) \cap V_C = \{u_k, u_i\}$ and $N(V_L) \cap V_C = \{u_i, u_j, u_k\}$.

Since $|V_C| \geq 15$, we may without loss of generality assume that V_{P_1} contains at least five vertices. Then we can choose $u_r \in V_{P_1}$ such that $u_{r-1} \neq u_i$ and $u_{r+1} \neq u_j$. We need to consider four subcases.

Case 3a. Vertex u_r is not adjacent to any components in $\mathcal{K}_1$, or u_r is adjacent to the vertex set of exactly one component $K \in \mathcal{K}_1$ and this component K has $N(V_K) \cap V_C \not\subseteq \{u_r, u_k\}$.

Let $x_2 \in V_{K_2}$ be a neighbor of u_k and let $y_2 \in V_{K_2}$ be a neighbor of v. We note that $x_2 = y_2$ is possible. Let Q_2 be a shortest path from x_2 to y_2 in K_2. Let $x_3 \in V_{K_3}$ be a neighbor of u_k and let $y_3 \in V_{K_3}$ be a neighbor of v. We note that $x_3 = y_3$ is possible. Let Q_3 be a shortest path from x_3 to y_3 in K_3.

We define the (chordless) cycle $C' = x_2 \overrightarrow{Q_2} y_2 v y_3 \overleftarrow{Q_3} x_3 u_k x_2$, and we claim that (C', u_r) is an H_9-contraction pair of G. By definition of C', C and our assumptions on K_2, K_3, we find that $N(u_r) \cap V_{C'} = \emptyset$. We show that $N(u_r)$ is in the component F of $G[V \backslash (V_{C'} \cup \{u_r\})]$ to which u_{r-1} belongs.

Since C is chordless, vertices u_{r-1} and u_{r+1} are the only neighbors of u_r on C. Consider u_{r+1}. By definition of K_1, the subgraph $G[V \backslash (V_{C'} \cup \{u_r\})]$ contains a path that starts with the subpath $u_{r+1} \overrightarrow{C} u_j$, then uses one or more vertices in K_1, and ends with the subpath $u_i \overrightarrow{C} u_{r-1}$. Hence, vertex u_{r+1} is in F.

Let w be a neighbor of u_r not on C. Suppose w is in some component $K' \in \mathcal{K}_2$. Since G is 3-connected, we find that $V_{K'}$ is adjacent to at least three vertices on C. One of these vertices is not in $\{u_r, u_k\}$. We denote this vertex by u_q. Then $G[V_C \backslash \{u_k, u_r\}] \subset G[V \backslash (V_{C'} \cup \{u_r\})]$ either contains a path from u_q to u_{r-1}, or else a path from u_q to u_{r+1}. Since $\{u_{r-1}, u_{r+1}\} \subset V_F$, we find, in both cases, that u_q, and hence w, is in F.

Suppose w is in some component in $\mathcal{K}_1$. Due to the subcase assumption, vertex w is in a component $K \in \mathcal{K}_1$ with $N(V_K) \cap V_C \not\subseteq \{u_r, u_k\}$. Since V_K is adjacent to at least two vertices on C, vertex set V_K is adjacent to some vertex $u_s \in V_C \backslash \{u_r, u_k\}$. Then subgraph $G[V_C \backslash \{u_k, u_r\}] \subset G[V \backslash (V_{C'} \cup \{u_r\})]$ either contains a path from u_s to u_{r-1}, or else a path from u_s to u_{r+1}. Since $\{u_{r-1}, u_{r+1}\} \subset V_F$ we find, in both cases, that u_q, and hence w, is in F. We conclude that $N(u_r)$ is in F.

Case 3b. Vertex u_r is adjacent to the vertex set of exactly one component $K \in \mathcal{K}_1$ and K has $N(V_K) \cap V_C \subseteq \{u_r, u_k\}$, and furthermore, u_{r+1} is *not* adjacent to a component $K' \in \mathcal{K}$ with a neighbor set $N(V_{K'})$ that contains a vertex on $u_{r+2}\overrightarrow{C}u_j$.

We first note that $K \notin \{K_1, K_2, K_3\}$. Furthermore, since any component in $\mathcal{K}_1$ has at least two neighbors in C, we find that $N(V_K) \cap V_C = \{u_r, u_k\}$.

Let $x_1 \in V_{K_1}$ be a neighbor of u_j and let $y_1 \in V_{K_1}$ be a neighbor of v. We note that $x_1 = y_1$ is possible. Let Q_1 be a a shortest path from x_1 to y_1 in K_1. Let $x_2 \in V_{K_2}$ be a neighbor of u_j and let $y_2 \in V_{K_2}$ be a neighbor of v. We note that $x_2 = y_2$ is possible. Let Q_2 be a shortest path from x_2 to y_2 in K_2.

We define the (chordless) cycle $C' = x_1\overrightarrow{Q_1}y_1vy_2\overleftarrow{Q_2}x_2u_jx_1$, and we claim that (C', u_r) is an H_9-contraction pair of G. By definition of C', C and our assumptions on K_1, K_2, we note that $N(u_r) \cap V_{C'} = \emptyset$. We show that $N(u_r)$ is in the component F of $G[V \backslash (V_{C'} \cup \{u_r\})]$ to which u_{r-1} belongs.

Since C is chordless, vertices u_{r-1} and u_{r+1} are the only neighbors of u_r on C. Consider u_{r+1}. Since G is 3-connected, vertex u_{r+1} has a neighbor in some component $K^* \in \mathcal{K}$,. We observe that $K^* \notin \{K, K_1, K_2, K_3\}$. The vertex set of K^* is adjacent to at least two vertices of V_C. Let vertex $z^* \in V_{K^*}$ have a neighbor $u_s \neq u_{r+1}$ on C. Due to our subcase assumption, we find that u_s is on the path $u_{j+1}\overrightarrow{C}u_{r-1}$, which is a path in $G[V \backslash (V_{C'} \cup \{u_r\})]$. Hence, u_{r+1} is in F.

Let w be a neighbor of u_r not on C. There are two cases: either w is in some component in $\mathcal{K}_1$, and then w must be in K, or w is in some component $L' \in \mathcal{K}_2$.

Suppose w is in K. Recall that $N(V_K) \cap V_C = \{u_r, u_k\}$. Then component K contains a vertex w' that is adjacent to u_k. We note that $w' = w$ is possible. Let P' be a path in K from w to w'. Then $G[V \backslash (V_{C'} \cup \{u_r\})]$ contains the path $w\overrightarrow{P'}w'u_k\overrightarrow{C}u_{r-1}$. This means that w is in F.

Suppose w is in some component $L' \in \mathcal{K}_2$. We observe that $L' \neq L$. Recall that $V_{L'}$ must be adjacent to at least three vertices on C. One of these vertices is not in $\{u_r, u_j\}$. We denote this vertex by u_s. Then $G[V_C \backslash \{u_j, u_r\}] \subset G[V \backslash (V_{C'} \cup \{u_r\})]$ either contains a path from u_s to u_{r-1}, or else a path from u_s to u_{r+1}. Since $\{u_{r-1}, u_{r+1}\} \subset V_F$ we find in both cases that u_s is in F. Since L' is adjacent to u_s, we then find that w is in F.

From the above, we conclude that $N(u_r)$ is in F. So (C', u_r) is indeed an H_9-contraction pair of G.

Case 3c. Vertex u_{r+1} is adjacent to some component $K' \in \mathcal{K}$ with a neighbor set $N(V_{K'})$ that contains a vertex on $u_{r+2}\overrightarrow{C}u_j$.

Let $z_1 \in V_{K'}$ be a neighbor of u_{r+1} and let $z_2 \in V_{K'}$ be adjacent to a vertex u_q on $u_{r+2}\overrightarrow{C}u_j$. We note that $z_1 = z_2$ is possible. We also note that $K' \neq K_1$ and $K' \neq K_3$. We let P be a path from z_1 to z_2 in K'. We define the cycle $C' = z_2\overleftarrow{P}z_1u_{r+1}\overrightarrow{C}u_qz_2$.

Let w be a non-cutvertex of K_3. We claim that (C', w) is an H_9-contraction pair of G. By definition of C' and our assumption on K_3, we find that $N(w) \cap V_{C'} = \emptyset$. We show that $N(w)$ is in the component F of $G[V \backslash (V_{C'} \cup \{w\})]$ to which u_i belongs.

Clearly, P_3 is a subgraph of F. Let t be a neighbor of w. If t is on C, then t is on P_3 and hence t is on F. Suppose t is in K_3. Since G is 3-connected and $|V_{K_3}| \geq 2$, component K_3 contains at least two different vertices w', w^* that are adjacent to $P_3 \subset F$. At least one of these vertices, say w', is not equal to w. Since vertex w is a non-cutvertex in K_3, there exists a path from t to w' in K_3 that does not use w. Hence, also in this case, t is in F. Suppose $t = v$. Since $K_1 \in \mathcal{K}_1$, vertex v is adjacent to a vertex t' in K_1. Recall that K_1 is adjacent to u_i. Then we find that v is in F. Hence, we conclude that $N(w)$ is in F.

Case 3d. Vertex u_r is adjacent to at least two components $K, K' \in \mathcal{K}_1$.

Then $G[V_K \cup V_{K'} \cup \{u_r, v\}]$ contains a cycle C' (which uses both u_r and v). Let w be a non-cutvertex in L. With an eye on Remark 1, we claim that (C', w) is an H_9-contraction pair of G. By definition of C' and L, we find that $N(w) \cap V_{C'} = \emptyset$. We show that $N(w)$ is in the component F of $G[V \backslash (V_{C'} \cup \{w\})]$ to which u_i belongs.

Let t be a neighbor of w. Suppose t is on C. Since we assume that $N(V_L) \cap V_C = \{u_i, u_j, u_k\}$ we find that $t \in \{u_i, u_j, u_k\}$. Then vertex t is connected to u_i in $G[V \backslash (V_{C'} \cup \{w\})]$ via the path $t\overrightarrow{C}u_i$. Suppose t is in L, so $|V_L| \geq 2$. If t is adjacent to V_C, then t is in F. Otherwise, since G is 3-connected, there exists a vertex $t' \neq w$ in L adjacent to V_C. Since vertex w is a non-cutvertex in L, component L contains a path from t to t' that does not use w. This means that t is F. Hence, we conclude that $N(w)$ is in F.

This finishes the proof of Lemma 3.6. □

From now on, we may assume that $G[V \backslash V_C]$ is connected for any pseudo-pair (C, v) of a 3-connected graph G. The next lemma provides us with information on the neighbors of C.

Lemma 3.7 *Let (C, v) be a pseudo-pair of a 3-connected graph $G = (V, E)$ with $C = u_1 u_2 \dots u_p u_1$ for some $p \geq 15$ such that $G[V \backslash V_C]$ is connected. If C contains a vertex not adjacent to the vertex set of some component in $G[V \backslash (V_C \cup \{v\})]$, then it is possible to find an H_9-contraction pair in polynomial time.*

Proof: Let $\mathcal{K}$ denote the set of components of $G[V \backslash (V_C \cup \{v\})]$. Then, due to our assumption that $G[V \backslash V_C]$ is connected, every component in $\mathcal{K}$ contains a vertex adjacent to v. Since G is 3-connected, the vertex set of any component in $\mathcal{K}$ is adjacent to at least two vertices on C. If $|\mathcal{K}| = 1$, then (C, v) is an H_9-contraction pair and we are done. So assume $|\mathcal{K}| \geq 2$. We first show the following claim.

Claim 1. If there exist a component $K \in \mathcal{K}$ and a path $P = u_{i_1} u_{i_2} u_{i_3} \subset C$ such that V_K and V_P are not adjacent, then it is possible to find an H_9-contraction pair of G in polynomial time.

We prove Claim 1 as follows. Let K be a component of $G[V \backslash (V_C \cup \{v\})]$ such that V_K is not adjacent to a path of three vertices of C, say K is not adjacent to $\{u_1, u_2, u_3\}$. We will show how to identify an H_9-contraction pair of G. It is easy to check that this can be done in polynomial time.

Let u_i and u_j be two neighbors of V_K on C. We assume without loss of generality that u_1, u_2, u_3 are on $u_i\overrightarrow{C}u_j$. Let $x \in V_K$ be adjacent to u_i and let $y \in V_K$ be adjacent to u_j. We note that $x = y$ is possible. Let P be a path from x to y in K. We define $C' = x\overrightarrow{P}yu_j\overleftarrow{C}u_ix$. With an eye on Remark 1, we claim that (C', u_2) is an H_9-contraction pair of G. By definition of C' and C, we note that $N(u_2) \cap V_{C'} = \emptyset$. We show that $N(u_2)$ is in the component F of $G[V \backslash (V_{C'} \cup \{u_2\})]$ to which v belongs.

Since C is chordless, vertices u_1 and u_3 are the only neighbors of u_2 that are on C. We first consider u_1. Since G is 3-connected, vertex u_1 has a neighbor z in some component $K' \in \mathcal{K} \backslash \{K\}$. Because $G[V \backslash V_C]$ is connected, K' contains a vertex adjacent to v. We then find that u_1 is in F. In the same way we prove that u_3 is in F. Any other neighbor z of u_2 is in a component in $\mathcal{K} \backslash \{K\}$. Let z be in $K^* \in \mathcal{K} \backslash \{K\}$. Then z is connected to v via a path in $V_{K^*} \cup \{v\}$, again because $G[V \backslash V_C]$ is connected. Hence $N(u_2)$ is in F. This finishes the proof of Claim 1.

From now on we may assume that the situation in Claim 1 does not occur (as otherwise we are done). So all components in $\mathcal{K}$ are adjacent to any set of three consecutive vertices on C. We will now prove the complete statement of Lemma 3.7. Suppose u is a vertex on C that is not adjacent to V_K for some $K \in \mathcal{K}$. We will show how to identify an H_9-contraction pair of G. It is easy to check that this can be done in polynomial time.

Cycle C has at least $p \geq 15$ vertices. Then we can choose a path $P \subset C$ on seven vertices, say without loss of generality, $P = u_1 u_2 \dots u_7$ such that $u_4 = u$, and $G[V_C \backslash V_P]$ contains a path $P' = u'_1 u'_2 \dots u'_6$ on at least six vertices. Recall that we assume that the situation under Claim 1 does not occur. So K contains a vertex x that has a neighbor $u'_i \in \{u'_1, u'_2, u'_3\}$ and K contains a vertex y that has a neighbor $u'_j \in \{u'_4, u'_5, u'_6\}$. We note that $x = y$ is possible. Let Q be a path from x to y in K. Then we define $C' = x\overrightarrow{Q}yu'_j\overleftarrow{P'}u'_i x$.

Keeping Remark 1 in mind we claim that (C', u) is an H_9-contraction pair. By definition of C' and C, we note that $N(u) \cap V_{C'} = \emptyset$. We show that $N(u)$ is in the component F of $G[V \backslash (V_{C'} \cup \{u\})]$ to which v belongs.

Any neighbor y of u not on C is in a component $K' \in \mathcal{K} \backslash \{K\}$. Since $G[V \backslash V_C]$ is connected, v is adjacent to $V_{K'}$. This implies that y is in F. As mentioned above, we assume that the situation under Claim 1 does not occur. Hence, any component in $\mathcal{K}$ has at least one neighbor in $\{u_1, u_2, u_3\}$ and at least one neighbor in $\{u_5, u_6, u_7\}$. Since $\mathcal{K}$ contains at least one other component besides K, we use our assumption that $G[V \backslash V_C]$ is connected again to conclude that also u_3 and u_5, which are the only neighbors of u on C, are in F. This finishes the proof of Lemma 3.7. □

From now on, we may assume that for any pseudo-pair (C, v) of a 3-connected graph G not only $G[V \backslash V_C]$ is connected, but that also all vertices of C have a neighbor in every component of $G[V \backslash (V_C \cup \{v\})]$. We say that a block L of a graph K is a *leaf-block* if L contains at most one cutvertex of K. Note that a leaf-block in a tree consists of a leaf together with its neighbor. We will show that we may assume that every component in $G[V \backslash (V_C \cup \{v\})]$ contains a vertex that is adjacent to at least two neighbors in C. First, in Lemma 3.8, we show that for components that are not trees we can even prove something stronger, namely that all its leaf-blocks contain a non-cutvertex (of the whole component) adjacent to at least two vertices on C. We need this extra information later on. In Lemma 3.9, we prove that also a component that is a tree contains a vertex that has two neighbors on C.

Lemma 3.8 *Let (C, v) be a pseudo-pair of a 3-connected graph $G = (V, E)$ with $C = u_1 u_2 \dots u_p u_1$ for some $p \geq 15$ such that $G[V \backslash V_C]$ is connected and such that all vertices of C have a neighbor in every component of $G[V \backslash (V_C \cup \{v\})]$. Let L be a leaf-block of a component K of $G[V \backslash (V_C \cup \{v\})]$. If K is not a tree and the vertices of L that are non-cutvertices of K have at most one neighbor in C, then it is possible to find an H_9-contraction pair of G in polynomial time.*

Proof: Let $\mathcal{K}$ denote the set of components of $G[V \backslash (V_C \cup \{v\})]$. If $|\mathcal{K}| = 1$, then (C, v) is an H_9-contraction pair and we are done. So assume $|\mathcal{K}| \geq 2$. Let K be a component in $\mathcal{K}$ that is not a tree. Let L be a leaf-block of K such that all vertices of L that are non-cutvertices of K are adjacent to at most one vertex on C. We will show how to identify an H_9-contraction pair of G. It is easy to check that this can be done in polynomial time.

We distinguish three cases based on the size of V_L.

Case 1. $|V_L| = 1$.

Then $V_K = V_L = \{x\}$ for some $x \in V \backslash (V_C \cup \{v\})$ meaning that K is a tree. So L contains at least two vertices.

Case 2. $|V_L| = 2$.

Then L consists of two adjacent vertices x, y. Since K is not a tree, only one of those vertices, say x, is a non-cutvertex of K. Since G is 3-connected and we assume that all vertices of L that are non-cutvertices of K have at most one neighbor on C, vertex x has a unique neighbor u_i on C, and x is adjacent to v as well.

Since K is not a tree, component K contains a block B that is not an edge. By definition, B is 2-connected. This implies that B contains a (chordless) cycle C'. Recall that $\mathcal{K}$ contains at least two components. Let x' be a non-cutvertex of a component $K' \in \mathcal{K} \backslash \{K\}$.

We claim that (C', x') is an H_9-contraction pair. By definition of C', we find that $N(x') \cap V_{C'} = \emptyset$. We show that $N(x')$ is in the component F of $G[V \backslash (V_{C'} \cup \{x'\})]$ to which v belongs.

We first note that any vertex $u_j \in V_C$ is in F due to the path $P = u_j \overrightarrow{C} u_i x v$. This means that any neighbors of x' on C are in F. Suppose y' is a neighbor of x' in K'. Since G is 3-connected, component K' contains a vertex $z' \neq x'$ that has a neighbor u_h on $C \subset F$. Since x' is a non-cutvertex of K', there exists a path from y' to z' in K' that does not use x'. Then we find that y' is in F. Hence $N(x')$ is in F.

Case 3. $|V_L| \geq 3$.

Then L contains two adjacent vertices x and y that are both non-cutvertices of K in such a way that $G[V_K \backslash \{x, y\}]$ is connected. This can be seen as follows. Let $x \in V_L$ be a non-cutvertex of K. Suppose the only neighbor of x in K is a cutvertex of K. Then V_L would consist of exactly two vertices. So let x be adjacent to a vertex $y \in V_L$ that is not a cutvertex of K. Suppose $G[V_K \backslash \{x, y\}]$ is not connected. Then $\{x, y\}$ is a two-vertex cut of L. Let F be a component in $G[V_L \backslash \{x, y\}]$ that does not contain a cutvertex of K. If $|V_F| \geq 2$ we take two vertices of F instead of x, y and check whether they are an appropriate choice. Suppose $V_F = \{z\}$. Then z is adjacent to $\{x, y\}$. As a matter of fact $N(z) \cap V_K = \{x, y\}$; otherwise x or y would be a cutvertex of K. We choose x, z instead of x, y. Suppose $G[V_K \backslash \{x, z\}]$ is not connected. Then x is a cutvertex of K, but this would contradict our choice of x. Hence we may indeed assume there exist two adjacent vertices x, y that are both non-cutvertices of K such that $G[V_K \backslash \{x, y\}]$ is connected.

Case 3a. Both x and y are not adjacent to V_C.

Since x is not a cutvertex of K, all neighbors of x in K are in the same component of $G[V_K \backslash \{x\}]$. Since y is not a cutvertex of K, all neighbors of y in K are in the same component of $G[V_K \backslash \{y\}]$. Then the following statements are easy to see. If y is not adjacent to v, then (C, y) is an H_9-contraction pair of G. If x is not adjacent to v, then (C, x) is an H_9-contraction pair of G. If both $[v, x]$ and $[v, y]$ are edges in G, then (C, x) and (C, y) are both H_9-contraction pairs of G.

Case 3b. Only one of the vertices x, y is adjacent to V_C.

We assume without loss of generality that x has a unique neighbor u_r on C. Since y is not a cutvertex of K, all neighbors of y in K are in the same component of $G[V_K \backslash \{y\}]$. If v is not adjacent to y, we then find that (C, y) is an H_9-contraction pair of G. So we assume that $[v, y] \in E$.

If v is adjacent to some vertex in $V_K\backslash\{y\}$, we again find that (C, y) is an H_9-contraction pair of G. Suppose y is the only vertex in K that is adjacent to v.

Since L is 2-connected and $|V_L| \geq 3$, block L contains a (chordless) cycle C'. Let x' be a non-cutvertex of a component $K' \in \mathcal{K}\backslash\{K\}$. We claim that (C', x') is an H_9-contraction pair of G. By definition of C' we find that $N(x') \cap V_{C'} = \emptyset$. We show that $N(x')$ is in the component F of $G[V\backslash(V_{C'} \cup \{x'\})]$ to which V_C belongs.

Let y' be a neighbor of x' not on C. Suppose y' is in K'. Then $|V_{K'}| \geq 2$ and, since G is 3-connected, component K' contains at least two vertices z', z^* that have a neighbor on C. At least one of them, say z', is not equal to x'. Since x' is not a cutvertex of K', there exists a path from y' to z' in $G[V_{K'}\backslash\{x'\}]$. This means that y' is in F.

Suppose v is a neighbor of x'. If there exists a vertex in $V_{K'}\backslash\{x'\} \subset V_F$ that is adjacent to v, then v is in F. Suppose the only vertex in K' that is adjacent to v is x'. Since G is 3-connected, vertex v must be adjacent to some vertex $w \notin \{x', y\}$. Since v is neither a neighbor of $V_K\backslash\{y\}$ nor a neighbor of $V_{K'}\backslash\{x'\}$, vertex w is in some component $K^* \in \mathcal{K}\backslash\{K, K'\}$. Since every component of $\mathcal{K}$ is adjacent to C, we find that v is in F. Hence we can conclude that $N(x')$ is in F.

Case 3c Both vertices x, y are adjacent to V_C.

Let u_r be the unique neighbor of x on C, and let u_s be the unique neighbor of y on C. We note that $u_r = u_s$ is possible. We construct a (chordless) cycle C' that consists of the vertices x, y together with the vertices of a path P between u_r and u_s on C. Since $|V_C| \geq 15$, we can choose P in such a way that $G[V_C\backslash V_P]$ contains a path Q on five vertices. We assume without loss of generality that $Q = u_1u_2 \dots u_5$.

We claim that (C', u_2) is an H_9-contraction pair of G. By definition of C' and C, we note that $N(u_2) \cap V_{C'} = \emptyset$. We show that $N(u_2)$ is in the component F of $G[V\backslash(V_{C'} \cup \{u_2\})]$ to which the vertex u_3 belongs.

Since C is chordless, vertices u_1 and u_3 are the only neighbors of u_2 that are on C. Since G is 3-connected, vertex u_1 has a neighbor z in some component $K' \in \mathcal{K}$. As we assume in the statement of the lemma, $V_{K'}$ contains a vertex x' that is adjacent to u_3. If $K' \neq K$, we immediately find that u_1 is in F. Suppose $K' = K$. Then x' can neither be equal to x nor to y. Since $G[V_K\backslash\{x, y\}]$ is connected, subgraph $G[V_K\backslash\{x, y\}]$ contains a path from z to x'. Then, also in this case, we find that u_1 is in F. We use the same arguments to prove that all other neighbors of u_2 belong to F as well. Hence, $N(u_2)$ is in F. This finishes the proof of Lemma 3.8. □

Lemma 3.9 *Let (C, v) be a pseudo-pair of a 3-connected graph $G = (V, E)$ with $C = u_1u_2 \dots u_pu_1$ for some $p \geq 15$ such that $G[V\backslash V_C]$ is connected and such that all vertices of C have a neighbor in every component of $G[V\backslash(V_C \cup \{v\})]$. If there exists a component of $G[V\backslash(V_C \cup \{v\})]$ with no vertex adjacent to at least two vertices on C, then it is possible to find an H_9-contraction pair of G in polynomial time.*

Proof: Let $\mathcal{K}$ denote the set of components of $G[V\backslash(V_C \cup \{v\})]$. If $|\mathcal{K}| = 1$, then (C, v) is an H_9-contraction pair and we are done. So assume $|\mathcal{K}| \geq 2$. Suppose all vertices of component $T \in \mathcal{K}$ are adjacent to at most one vertex on C. We show how to identify an H_9-contraction pair of G. It is easy to check that this can be done in polynomial time.

First, if T is not a tree then we are done due to Lemma 3.8. So we may assume that T is a tree. Suppose T consists of a single vertex w. Since G is 3-connected, vertex w must have at least two neighbors on C. So T contains at least two vertices.

Let x and z be leaves of T. Since G is 3-connected, x has a neighbor u_i on C. Due to our assumption, vertex u_i is the only neighbor of x on C. Let $y \neq x$ be the first vertex on the unique path P from x to z in T that has a (unique) neighbor u_j on C. Since z has a neighbor on C, such a vertex y exists. In fact, $y = z$ is possible. We also note that $u_i = u_j$ are possible.

We construct a (chordless) cycle C' that consists of the vertices from the path $x\overrightarrow{P}y$ together with the vertices of a path P' from u_i to u_j on C. Since $|V_C| \geq 15$, we can choose P' in such a way that $G[V_C \backslash V_{P'}]$ contains a path on three vertices. Without loss of generality we assume that these three vertices are u_1, u_2, u_3.

We claim that (C', u_2) is an H_9-contraction pair of G. By definition of C' and C, we find that $N(u_2) \cap V_{C'} = \emptyset$. We show that $N(u_2)$ is in the component F of $G[V \backslash (V_{C'} \cup \{u_2\})]$ to which v belongs.

Since C is chordless, vertices u_1 and u_3 are the only neighbors of u_2 that are on C. We first consider u_1. Since G is 3-connected, vertex u_1 has a neighbor z' in some component $K' \in \mathcal{K}$. Suppose $K' \neq T$. Then, since $G[V \backslash V_C]$ is connected, v has a neighbor in K'. Then u_1 is in F. Suppose $K' = T$. By definition of y, vertex z' is not on $x\overrightarrow{P}y$. Then there exists a path from z' to a leaf $z'' \neq x$ of T that does not contain a vertex of $x\overrightarrow{P}y$. Since G is 3-connected and all vertices of T have at most one neighbor in C, leaf z'' must be adjacent to v. We find that, also in this case, vertex u_1 is in F. In exactly the same way we prove that u_3 is in F. We furthermore note that all other neighbors of u_2, which are not on C, are still connected to v, because $G[V \backslash V_C]$ is connected. Hence $N(u_2)$ is in F. This finishes the proof of Lemma 3.9. □

As mentioned before, we may assume that for any pseudo-pair (C, v) of a 3-connected graph G, $G[V \backslash V_C]$ is connected and all vertices of C have a neighbor in every component of $G[V \backslash (V_C \cup \{v\})]$. In Lemma 3.9 we have shown that each component in $G[V \backslash (V_C \cup \{v\})]$ can be assumed to contain a vertex adjacent to at least two vertices on C. In order to get a complete pseudo-pair of G, we will now show that we may assume that every vertex in $V \backslash V_C$ that has at least two neighbors in C is adjacent to all vertices of C. First, in Lemma 3.10 we show that this is true for non-cutvertices of components in $G[V \backslash (V_C \cup \{v\})]$. Then, in Lemma 3.11, we give a proof for the remaining vertices in $V \backslash V_C$.

Lemma 3.10 *Let (C, v) be a pseudo-pair of a 3-connected graph $G = (V, E)$ with $C = u_1u_2 \dots u_pu_1$ for some $p \geq 15$ such that $G[V \backslash V_C]$ is connected and such that all vertices of C have a neighbor in every component of $G[V \backslash (V_C \cup \{v\})]$. Let x be a non-cutvertex of a component of $G[V \backslash (V_C \cup \{v\})]$. If x is adjacent to at least two vertices on C but not to all of them, then it is possible to find an H_9-contraction pair in polynomial time.*

Proof: Let $\mathcal{K}$ denote the set of components of $G[V \backslash (V_C \cup \{v\})]$. If $|\mathcal{K}| = 1$, then (C, v) is an H_9-contraction pair and we are done. So assume $|\mathcal{K}| \geq 2$. Let x be a non-cutvertex in a component $K \in \mathcal{K}$ that is adjacent to at least two vertices on C, but not to all vertices on C. We will show how to identify an H_9-contraction pair of G. It is easy to check that this can be done in polynomial time.

We need to distinguish two cases: either x is not adjacent to a set of three consecutive vertices on C, or x is adjacent to all sets of three consecutive vertices on C.

Case 1. Vertex x is not adjacent to the vertices of some path $P \subset C$ on three vertices.

We assume without loss of generality that $P = u_1u_2u_3$. Let u_r, u_s be neighbors of x on C such that $u_r\overrightarrow{C}u_s$ does not contain the vertices u_1, u_2, u_3. Let $C' = xu_r\overrightarrow{C}u_sx$.

Keeping Remark 1 in mind, we claim that (C', u_2) is an H_9-contraction pair of G. By definition of C' and C, we note that $N(u_2) \cap V_{C'} = \emptyset$. We show that all vertices of $N(u_2)$ are in the component F of $G[V \backslash (V_{C'} \cup \{u_2\})]$ to which the vertex u_1 belongs.

Since C is chordless, vertices u_1 and u_3 are the only neighbors of u_2 that are on C. In the statement of the lemma we assume that both u_1 and u_3 have a neighbor in K. Let y be the neighbor of u_1 in K and let y' be the neighbor of u_3 in K. We note that $y = y'$ is possible. Since x is adjacent neither to u_1 nor to u_3, vertices y and y' are not equal to x. Since x is a non-cutvertex of K, there exists a path P from y to y' in K. Hence u_3 is in F.

Let z be a neighbor of u_2 that is not on C. Suppose z is in K. Clearly, $z \neq x$. Since x is a non-cutvertex of K, there exists a path from z to y in K. Hence z is in F. Suppose z is in $K' \in \mathcal{K} \backslash \{K\}$. As we assume in the statement of the lemma, vertex u_1 has a neighbor z' in K' as well. Again we find that z is in F. Hence, we conclude that $N(u_2)$ is in F.

Case 2. Vertex x is adjacent to a vertex on every path $P \subset C$ on three vertices.

Since x is not adjacent to all vertices on C, there exists a vertex $u_h \in V_C$ with $[u_h, x] \notin E$. We need to consider two subcases.

Case 2a. There exists a vertex in $V_C \backslash \{u_h\}$ that has a neighbor in $V_K \backslash \{x\}$.

Then there exists a vertex $u_i \neq u_h$ on C such that u_i is the only vertex of $u_i \overrightarrow{C} u_{h-1}$ that is adjacent to $V_K \backslash \{x\}$. Let $R = u_{i+1} \overrightarrow{C} u_{h-2}$ and $R' = u_{h+2} \overrightarrow{C} u_{i-1}$. Note that R or R' may be empty. However, since $|V_C| \geq 15$, we find that $|V_R| \geq 2$ or else that $|V_{R'}| \geq 6$. If $|V_R| \geq 2$ we choose $P = R$ and else we choose $P = R'$. We claim that x is adjacent to two vertices u_r, u_s on P. This can be seen as follows. If $P = R$ (so $|V_R| \geq 2$) then vertex x is even adjacent to all vertices on P, due to the definition of R and our assumption (in the statement of the lemma) that every vertex on C is adjacent to at least one vertex of K. If $P = R'$ (so $|V_{R'}| \geq 6$) then x has at least two neighbors on P, due to our assumption that x is adjacent to every set of three consecutive vertices on C. We note that u_r, u_s are not adjacent to u_h. This is important for the rest of our proof of this subcase.

We assume $r < s$, and we define $C' = x u_r \overrightarrow{C} u_s x$. With an eye on Remark 1, we claim that (C', u_h) is an H_9-contraction pair of G. By definition of C' and C, we find that $N(u_h) \cap V_{C'} = \emptyset$. We show that all vertices of $N(u_h)$ are in the component F of $G[V \backslash (V_{C'} \cup \{u_h\})]$ to which vertex u_{h-1} belongs.

Since C is chordless, vertices u_{h-1} and u_{h+1} are the only neighbors of u_h that are on C. As we assume in the statement of the lemma, both u_{h-1} and u_{h+1} have a neighbor in some $K' \in \mathcal{K}$ with $K' \neq K$. Hence u_{h+1} in F. Therefore, all vertices in $V_C \setminus V_{C'}$ are in F (including u_i). Let z' be a neighbor of u_h that is not on C. Suppose z' is in K. Clearly, $z' \neq x$. Since x is a non-cutvertex of K, there exists a path from z' to z in K. Recall that z is adjacent to u_i. Since u_i is in F, we find that z' is in F. Suppose z' is in $K' \in \mathcal{K}$ with $K' \neq K$. By the statement of the lemma, vertex u_{h-1} has a neighbor z'' in K' as well. Then, again we find that z' is in F. Hence, we conclude that $N(u_h)$ is in F.

Case 2b. There does not exist a vertex in $V_C \backslash \{u_h\}$ that has a neighbor in $V_K \backslash \{x\}$.

As we assume in the statement of the lemma, u_h has a neighbor y in K. Since $u_h \notin N(x)$, we find that $y \neq x$. Suppose $V_K \backslash \{x\}$ is not adjacent to v. If $|V_K| \geq 3$, then any vertex $z \in V_K \backslash \{x, y\}$ is only connected to C via a path using x or u_h. Since G is 3-connected, this is not possible. We derive that $V_K = \{x, y\}$. However, since G is 3-connected, y must have a neighbor u_i in $V_C \backslash \{u_h\}$. This contradicts our assumption that such a vertex u_i does not exist. Hence, we may assume that $V_K \backslash \{x\}$ is adjacent to v.

Since $|V_C| \geq 15$, the path $P = u_{h+2} \overrightarrow{C} u_{h-2}$ has at least twelve vertices. Then, due to our assumption on $N(x) \cap V_C$, path P contains two vertices u_r, u_s that are adjacent to x. We assume $r < s$, and we define $C' = x u_r \overrightarrow{C} u_s x$.

With an eye on Remark 1, we claim that (C', u_h) is an H_9-contraction pair of G. By definition of C' and C, we note that $N(u_h) \cap V_{C'} = \emptyset$. We show that all vertices of $N(u_h)$ are in the component

F of $G[V\backslash(V_{C'} \cup \{u_h\})]$ to which vertex v belongs.

Since C is chordless, vertices u_{h-1} and u_{h+1} are the only neighbors of u_h that are on C. By the statement in the lemma, both u_{h-1} and u_{h+1} have a neighbor in some $K' \in \mathcal{K}$ with $K' \neq K$. Because $G[V\backslash V_C]$ is connected, $V_{K'}$ is adjacent to v. Hence u_{h-1}, u_{h+1} are in F. Let z' be a neighbor of u_h that is not on C. Suppose z' is in K. Since x is a non-cutvertex of K, the graph $G[V_K\backslash\{x\}]$, whose vertex set is adjacent to v, is connected. This means that z' is in F. Suppose z' is in $K' \in \mathcal{K}$ with $K' \neq K$. Because $G[V\backslash V_C]$ is connected, $V_{K'}$ is adjacent to v. Again we find that z' is in F. We conclude that $N(u_h)$ is in F. This finishes the proof of Lemma 3.10. □

Lemma 3.11 *Let (C, v) be a pseudo-pair of a 3-connected graph $G = (V, E)$ with $C = u_1u_2 \dots u_pu_1$ for some $p \geq 15$ such that $G[V\backslash V_C]$ is connected and such that all vertices of C have a neighbor in every component of $G[V\backslash(V_C \cup \{v\})]$. If there exists a vertex $x \in V\backslash V_C$ adjacent to at least two vertices on C but not to all of them, then it is possible to find an H_9-contraction pair in polynomial time.*

Proof: Let $\mathcal{K}$ denote the set of components of $G[V\backslash(V_C \cup \{x\})]$. If $|\mathcal{K}| = 1$, then (C, v) is an H_9-contraction pair and we are done. So assume $|\mathcal{K}| \geq 2$. Let $x \in V_K$ for some $K \in \mathcal{K}$ be adjacent to at least two vertices on C but not to all vertices of C. We will show how to identify an H_9-contraction pair of G. It is easy to check that this can be done in polynomial time.

If x is a non-cutvertex of a component of $G[V\backslash(V_C \cup \{v\})]$ then we are done due to Lemma 3.10. Suppose x is a cutvertex of $K \in \mathcal{K}$. We first prove the following claim that enables us to create a new pseudo-pair (which will turn out to be an H_9-contraction pair).

Claim 1. There exist vertices u_r, u_s, u_h with $\{u_r, u_s\} \subseteq N(x)$ and $u_h \notin N(x)$, such that u_h is on $u_{r+2}\overrightarrow{C}u_{s-2}$.

We note that we do allow u_{h-1}, u_{h+1} to be adjacent to x. We only want to make sure that these two vertices are not in $\{u_r, u_s\}$, because later we define a cycle C' with u_r, u_s on it, such that (C', u_h) turns out to be an H_9-contraction pair of G. We can choose u_r, u_s, u_h as follows.

Suppose x is adjacent to at most four vertices u_a, u_b, u_c, u_d on C. Since $|V_C| \geq 15$, we may without loss of generality assume that $u_a\overrightarrow{C}u_b$ contains at least five vertices, such that only vertices u_a and u_b are adjacent to x. Then we choose $u_r = u_a, u_s = u_b$ and $u_h = u_{a+2}$.

Suppose x is adjacent to all but one vertices on C, say x is not adjacent to u_1. Then we choose $u_h = u_1$, $u_r = u_{p-1}$ and $u_s = u_3$.

Suppose x is adjacent to all but two vertices on C, say x is not adjacent to u_1 and u_j for some $j \geq 2$. If $j \notin \{3, p-1\}$ we choose $u_h = u_1$, $u_r = u_{p-1}$ and $u_s = u_3$. Otherwise, we choose $u_h = u_1$, $u_r = u_{p-2}$ and $u_s = u_4$. We can do this, because $p \geq 15$.

Suppose x is adjacent to at least five vertices u_a, u_b, u_c, u_d, u_e with $a < b < c < d < e$ and x is not adjacent to at least three vertices on C. Suppose $u_b\overrightarrow{C}u_d$ contains a vertex not adjacent to x. Then we let u_h be that vertex, and we choose $u_r = u_a$ and $u_s = u_e$. Suppose $u_b\overrightarrow{C}u_d$ only contains vertices adjacent to x. Then we may without loss of generality assume that $u_b\overrightarrow{C}u_d = u_bu_cu_d$. This implies that $u_e\overrightarrow{C}u_a$ contains at least twelve vertices, and three of them are not adjacent to x. We let u_h be the second vertex on the path $u_e\overrightarrow{C}u_a$ that is not adjacent to x. We choose $u_r = u_d$ and $u_s = u_b$. This finishes the proof of Claim 1.

Let u_r, u_s, u_h be the vertices of Claim 1. We define $C' = xu_r\overleftarrow{C}u_sx$ and, having Remark 1 in mind, claim that (C', u_h) is an H_9-contraction pair of G. By definition of C' and C, we find that $N(u_h) \cap V_{C'} = \emptyset$. We show that all vertices of $N(u_h)$ are in the component F of $G[V\backslash(V_{C'} \cup \{u_h\})]$ to which u_{h-1} belongs.

Since C is chordless, vertices u_{h-1} and u_{h+1} are the only neighbors of u_h that are on C. Let K' be a component in $\mathcal{K}\backslash\{K\}$. As we assume in the statement of the lemma, both u_{h-1} and u_{h+1} have a neighbor in K'. Hence u_{h+1} is in F.

Let z be a neighbor of u_h on K. Then $G[V_K\backslash\{x\}]$ contains a path from z to some vertex y of a leaf-block L of K such that y is a non-cutvertex of K. Suppose y is adjacent to at least two vertices on C. By Lemma 3.10, we may assume that y is adjacent to all vertices on C (otherwise we are done). As a consequence, y is adjacent to u_{h-1}. Then z is in F. Suppose y is adjacent to at most one vertex on C, and suppose we can not replace y by some vertex in L that is a non-cutvertex of K and that is adjacent to at least two vertices on C. So all vertices in L that are non-cutvertices in K are adjacent to at most one vertex on C. Then, by Lemma 3.8, we may assume that K is a tree and y is a leaf of K (otherwise we are done). Since G is 3-connected, y must be adjacent to v. Since $G[V\backslash V_C]$ is connected, v is adjacent to a component $K^* \in \mathcal{K}\backslash\{K\}$ and V_{K^*} is adjacent to u_{h-1} by the statement of the lemma. This way we find that vertex z is in F.

Let z' be a neighbor of u_h on a component $K' \in \mathcal{K}\backslash\{K\}$. By the statement of the lemma, u_{h-1} has a neighbor in K'. Then z' is in F.

From the above, we conclude that $N(u_h)$ is in F. This finishes the proof of Lemma 3.11. □

We can now finally prove that a sufficiently large pseudo-pair of a 3-connected graph may be assumed to be complete.

Lemma 3.12 *Let (C, v) be a pseudo-pair of a 3-connected graph $G = (V, E)$ with $C = u_1u_2 \dots u_pu_1$ for some $p \geq 15$. If (C, v) is not complete then it is possible to find an H_9-contraction pair of G in polynomial time.*

Proof: Suppose (C, v) is not a complete pseudo-pair of G. We will show how to identify an H_9-contraction pair of G. It is easy to check that this can be done in polynomial time.

First, we may assume that $G[V\backslash V_C]$ is connected. Otherwise we are done by Lemma 3.6. Let $\mathcal{K}$ denote the set of components of $G[V\backslash(V_C \cup \{x\})]$. If $|\mathcal{K}| = 1$, then (C, v) is an H_9-contraction pair and we are done. So assume $|\mathcal{K}| \geq 2$. If there exists a component in $\mathcal{K}$ with no vertex adjacent to at least two vertices on C then we are done due to Lemma 3.9. So we may assume that every component $K \in \mathcal{K}$ has a vertex y adjacent to at least two vertices on C. We may even assume that such a y is adjacent to all vertices on C. Otherwise we are done due to Lemma 3.11.

Recall that (C, v) would have been a complete pseudo-pair of G if

(i) every vertex in $V\backslash(V_C \cup \{v\})$ is adjacent to v, so $N(v) = V\backslash(V_C \cup \{v\}$;

(ii) every vertex in $V\backslash(V_C \cup \{v\})$ is adjacent to all vertices on C.

So (C, v) fails condition (i) or (ii).

Case 1. (C, v) fails (ii).

Let x be a vertex of a component K that is not adjacent to all vertices on C. Then we may assume that x is adjacent to at most one vertex of C. Otherwise we are done due to Lemma 3.11. So we assume without loss of generality that x is not adjacent to $V_C\backslash\{u_1\}$ (while x is possibly adjacent to u_1). Let K' be a component in $\mathcal{K}\backslash\{K\}$. As we argued above, K' contains a vertex y' adjacent to all vertices on C.

We define $C' = y'u_3u_4y'$, and claim that (C', x) is an H_9-contraction pair of G. Obviously, $N(x) \cap V_{C'} = \emptyset$. Let u_2 belong to component F of $G[V\backslash(V_{C'} \cup \{x\})]$. We are done if we can show $N(x)$ belongs to F.

We first show $v \in V_F$. Since G is 3-connected, there exists a path P from v to u_2 not using the vertices x, y. Let u_i be the first vertex of P that is on C, i.e., $v\overrightarrow{P}u_i$ does not contain any other vertices of C except u_i. If $i \leq 2$ or $i \geq 5$, we define $P' = v\overrightarrow{P}u_i\overrightarrow{C}u_2$. Then v belongs to F, since P' is a subgraph of F. Suppose $i \in \{3, 4\}$. We define $P'' = v\overrightarrow{P}u_i\overleftarrow{C}u_2$. We replace C' by $C'' = y'u_5u_6y'$ in (C', x) in order to find $v \in V_F$ via P''. So we may assume without loss of generality that v belongs to F.

Now let w be a neighbor of x. If $w = v$, then w belongs to F as we just showed. If $w = u_1$, it is also clear that w belongs to F. Suppose $w \in V_K$. Then $G[V_K \backslash \{x\}]$ contains a path from w to a vertex z in a leaf-block L of K such that z is a non-cutvertex of K. Suppose K is not a tree. Due to Lemma 3.9, we may assume that z is adjacent to at least two vertices on C (as otherwise we are done). Due to Lemma 3.11, we then may even assume that z is adjacent to all vertices on C, including u_2 (as otherwise we are done). Then w is in F.

Suppose K is a tree. Then z is a leaf of K. If z is adjacent to at least two vertices on C, then we may assume that z is adjacent to all vertices on C including u_2, due to Lemma 3.11. Then w is in F. Otherwise, since G is 3-connected, z must be adjacent to v. Since $v \in V_F$, we find that z, and consequently w, belongs to F. This shows that (C', x) (or else (C'', x)) is an H_9-contraction pair of G.

Case 2. (C, v) fails (i) but does not fail (ii).

So we assume that all vertices in $V \backslash (V_C \cup \{v\})$ are adjacent to all vertices on C. Let $x \in V_K$ for some component $K \in \mathcal{K}$ such that x is not adjacent to v. Since x is adjacent to all vertices on C, we define $C' = xu_1u_2x$. Then $G[V \backslash (V_{C'} \cup \{v\})]$ is connected because all vertices in $V \backslash (V_C \cup \{v\})$ are adjacent to all vertices on C. Furthermore, v is not adjacent to $V_{C'}$. Hence we find that (C', v) is an H_9-contraction pair of G. This finishes the proof of Lemma 3.12. □

Step 6. Transform a complete pseudo-pair into an H_9-contraction pair

In the previous step we have shown how our algorithm found an H_9-contraction cycle of a 3-connected graph G with a "large" pseudo-pair that is not complete. So we are left to consider a 3-connected graph G with a complete pseudo-pair. In the next lemma we give a sufficient and necessary condition for a graph with complete pseudo-pair to be H_9-contractible.

Lemma 3.13 *Let (C, v) with $|V_C| \geq 15$ be a complete pseudo-pair of a 3-connected graph $G = (V, E)$ that is not an H_9-contraction pair of G. Then G has an H_9-contraction pair if and only if $G[N(v)]$ contains a cycle on at most $|N(v)| - 2$ vertices.*

Proof: Suppose G has an H_9-contraction pair (C', v'). If $v' = v$, then $C' = C$, and (C, v) would be an H_9-contraction pair. If $C' = C$, then $v' = v$, and (C, v) would be an H_9-contraction pair. So we find that $v' \neq v$ and $C' \neq C$.

Suppose v' is on C. Since (C, v) is a complete pseudo-pair of G, vertex v' is adjacent to all vertices in $N(v) = V \backslash (V_C \cup \{v\})$. Then the cycle C' does not contain any vertex of $N(v)$. Since $C' \neq C$, we find that v' can not be on C. Then v' must be in $N(v)$. This means that v' is adjacent to v and to all vertices on C. Hence C' only consists of vertices in $V \backslash (V_C \cup \{v\}) = N(v)$. Suppose $C' \cup \{v'\} = N(v)$. Then $v \in N(v')$ and $V_C \subset N(v')$ are in different components of $G[V \backslash (V_{C'} \cup \{v'\})]$. Hence, we conclude that C' is a cycle in $G[N(v)]$ that contains at most $|N(v)| - 2$ vertices.

For the other direction of the proof, suppose $G[N(v)]$ contains a cycle C' on at most $|N(v)| - 2$ vertices. Let $V_{C'}$ be in component K of $G[N(v)]$. Since (C, v) is not an H_9-contraction pair of G, there exists a vertex v' adjacent to v that is in some component K' of $G[N(v)]$ with $K' \neq K$.

We claim that (C', v') is an H_9-contraction pair of G. Since all the neighbors of v' are in $V_{K'} \cup V_C \cup \{v\}$, we find that $N(v') \cap V_{C'} = \emptyset$. We show that all neighbors of v' belong to the component F in $G[V \backslash (V_{C'} \cup \{v'\})]$ to which v belongs.

Let z be a neighbor of v' not equal to v. Suppose z is in K'. Then $z \in N(v)$. This means that z is in F. Suppose z is on C. Since C' contains at most $|N(v)| - 2$ vertices, there exists a vertex $w \in N(v) \backslash (V_{C'} \cup \{v'\})$. Vertex w is adjacent to z and adjacent to v. This means that z is in F. Hence, we conclude that $N(v')$ is in F. □

We are now ready to state the following theorem for 3-connected graphs.

Theorem 5 *Let G be a 3-connected graph. It is possible in polynomial time either to find H_9-witness sets of G, or else to conclude that G is not H_9-contractible.*

Proof: Let $G = (V, E)$ be a 3-connected graph. Due to Lemma 3.3 we can restrict ourselves to finding an H_9-contraction pair of G. Obtaining H_9-witness sets from an H_9-contraction pair in polynomial time is straightforward (see e.g. the proof of Lemma 3.3).

Due to Lemma 3.4 we can find in polynomial time an H_9-contraction pair (C, v) of G with $|V_C| \leq 14$, if G has such a pair. Otherwise, due to Lemma 3.5, we can, in polynomial time, either find a pseudo-pair (C', v') of G with $|V_{C'}| \geq 15$, or else conclude that G does not have an H_9-contraction pair.

Suppose we find a pseudo-pair (C', v') of G with $|V_{C'}| \geq 15$. It is easy to see that we can check in polynomial time whether (C', v') is complete. If (C', v') is not a complete pseudo-pair then, due to Lemma 3.12, we can find an H_9-contraction pair in polynomial time.

Suppose the pseudo-pair (C', v') is complete. Then by using Lemma 3.13, it remains to check in polynomial time whether $G[N(v)]$ contains a cycle C'' on at most $|N(v)| - 2$ vertices. This can be done as follows. We guess a pair of vertices x_1, x_2 in $G[N(v)]$ not to be on C''. We then apply the Breath-First-Search algorithm (BFS) in the graph $G[N(v) \backslash \{x_1, x_2\}]$. If the BFS algorithm finds a *back-edge* then we find a desired cycle. If not, then we take another pair x_3, x_4, and repeat the above procedure with the graph $G[N(v) \backslash \{x_3, x_4\}]$. □

Due to Corollary 4 and Theorem 5 we immediately obtain the main result of this section.

Corollary 6 *The H_9-CONTRACTIBILITY problem is solvable in polynomial time. Moreover, if a graph G is H_9-contractible, an H_9-witness structure of G can be found in polynomial time.*

4 Another tough polynomially solvable case

The graph W_k that is obtained from a cycle C on k vertices for some integer $k \geq 3$ by adding a new vertex adjacent to all vertices of C is called a *wheel* on $k+1$ vertices. Then the pattern graph $H_{10} = W_4$ (cf. Figure 1).

We denote the dominating vertex of a wheel W_k by x, and the other vertices by $y_1, \ldots y_k$ such that $[y_i, y_{i+1}] \in E_{W_k}$ for $1 \leq i \leq k-1$ and $[y_k, y_1] \in E_{W_k}$. If a graph $G = (V, E)$ is W_k-contractible with witness sets $W(x)$ and $W(y_1), \ldots, W(y_k)$ then we denote $X := W(x)$ and $Y_i := W(y_i)$ for $i = 1, \ldots, k$.

Unlike the wheel $W_3 = K_4$, wheels on more than four vertices are no longer complete graphs. This makes the correctness proof of a polynomial time algorithm for the W_4-CONTRACTIBILITY problem more complicated. Below we provide a polynomial time algorithm for the pattern graph $H_{10} = W_4$. Given a graph G, our algorithm either concludes that G is not W_4-contractible, or finds W_4-witness sets of G. In Section 4.1 we give an intuitive description of our algorithm, and in Section 4.2 we discuss it in detail.

4.1 Outline of the algorithm

We will act as follows.

Step 1. Increase the connectivity as much as possible

We first try to restrict ourselves to p-connected input graphs $G = (V, E)$ with p as high as possible. The intuitive reason behind this is that W_4-contractible graphs with a higher connectivity are expected to have easier to analyze witness structures than those with lower connectivity. We did not succeed in showing $p \geq 4$ but we could show that we may choose $p = 3$. First we show that $p \geq 2$ by using Lemma 2.1: if our input graph G is 1-connected then our algorithm only has to find a block of G that is W_4-contractible. So from now on we assume that G is 2-connected. In Lemma 4.1 we give a necessary and sufficient condition for G to be W_4-contractible. In Corollary 7 we show that our algorithm may break G into a polynomial number of smaller 3-connected parts that can be processed one by one.

Step 2. Decrease the search space of possible W_4-witness structures

As explained in Step 1, we now may assume that G is 3-connected. This has the following advantage: if G contains a chordless cycle $C = v_1v_2 \dots v_pv_1$ with $p \geq 4$ such that $G[V \backslash V_C]$ is connected, then G is W_4-contractible. This can be seen as follows. Since all vertices of G have at least degree three, any vertex on C has a neighbor in $G[V \backslash V_C]$. Hence, G is W_4-contractible with witness sets $Y_1 = \{v_i\}$ for $1 \leq i \leq 3$, $Y_4 = \{v_4, \dots, v_p\}$ and $X = V \backslash V_C$. This is why we call C a *W_4-contraction cycle* of G (see Figure 7 for an illustration of a W_4-contraction cycle C) . In Lemma 4.2 we show that having a W_4-contraction cycle is not only a sufficient but also a necessary condition for a 3-connected graph to be W_4-contractible. This structural result restricts the search space of possible witness structures considerably.

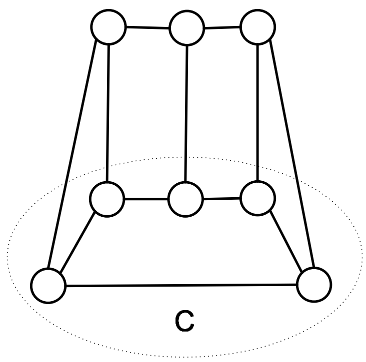

Figure 7: An example graph G with a W_4-contraction cycle C over five vertices of G.

Step 3. Find a chordless cycle on at least four vertices and do some easy checks

As explained in the previous step, in this stage our algorithm searches for a W_4-contraction cycle of G (although it sometimes finds W_4-witness sets directly). We observe that a chordless cycle C on at least four vertices is not necessarily a W_4-contraction cycle of G. It may happen that $G[V \backslash V_C]$

contains more than one component. However, to get started, any chordless cycle will do. If we find one that is not a W_4-contraction cycle, then we just perform a number of (efficient) checks on G with respect to C, in order to see if we can modify C into a W_4-contraction cycle. If not we move on to Step 4. Below we will explain the checks.

First, in Lemma 4.3, we show that we can find a chordless cycle on at least four vertices in polynomial time if G has such a cycle. Otherwise, if G does not have such a cycle, then G does not have a W_4-contraction cycle, and hence is not W_4-contractible. So assume G has a chordless cycle C on at least four vertices. Let $\mathcal{K}$ be the set of components of $G[V \backslash V_C]$. We show in Lemma 4.4 that G is W_4-contractible if one of the following cases is true:

(i) there exists a vertex $v \in V_C$ that is not adjacent to any vertex in some $K \in \mathcal{K}$;

(ii) $|\mathcal{K}| = 1$ or $|\mathcal{K}| \geq 3$;

(iii) there exists a component $K \in \mathcal{K}$ that contains two vertices u_1 and u_2 that are both adjacent to all vertices of C.

These three conditions are easy to check. Our algorithm stops if one of them is true. Then, due to Lemma 4.5, we can find a W_4-witness structure in polynomial time. Now suppose all three conditions are false. Then $|\mathcal{K}| = 2$, say $\mathcal{K} = \{K, K'\}$, so $V = V_C \cup V_K \cup V_{K'}$. We then say that C induces a *basic state* (C, K, K') of G. So, a basic state (C, K, K') of G has the following two properties.

(1) Every vertex in C has a neighbor in both K and K'.

(2) Both K and K' contain at most one vertex that is adjacent to all vertices of C.

Step 4. Decrease the size of one of the two remaining components

As explained in the previous step, by now we have a basic state (C, K, K') of G. If we can somehow completely eliminate K then we have found a W_4-contraction cycle, and we can stop. So, our next goal is to decrease the size of one of the components, say K, as much as possible. We can do this by considering chordless cycles of $G[V_C \cup V_K]$ that have at least one vertex of K (so they cannot be equal to C itself). Let D be such a cycle. First we check whether D induced a basic state. If not, then we are done as we explained in Step 3. In the other case, i.e., if we find a basic state (D, L, L'), then we show in Lemma 4.6 that L is a proper subgraph of K. Then, in Lemma 4.7, we show that we can repeat this argument again and again, in polynomial time, until we end in one of the following two situations: either we found a W_4-witness structure of G, or else we cannot decrease the first component of the basic state anymore (by using this technique). In the latter case, we say that the basic state is an *advanced state* of G (see Figure 8 for an illustration of an advanced state (C, K, K') where $K = \{u\}$). So an advanced state of G is a basic state with the extra property that the subgraph of G induced by the vertices in the cycle and in the component we tried to decrease does not contain a chordless cycle on at least four vertices with at least one vertex not on the cycle. We have to find a new technique that deals with advanced states.

Step 5. Detect a vertex adjacent to all vertices in the cycle and do an easy check

By now, we have an advanced state (C, K, K'). Since any advanced state is also a basic state, by definition, K contains at most one vertex adjacent to all vertices of C. In Lemma 4.8, we show that such a vertex u indeed exists, and we call u the *main vertex* of K. We call a chordless cycle

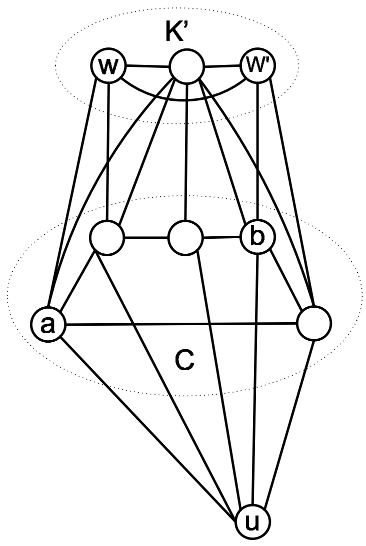

Figure 8: An example graph G with advanced state $(C, \{u\}, K')$, and a correct u-cycle $uaww'bu$.

D that has at least four vertices and that contains u, a *u-cycle* of G. In Lemma 4.9, we further analyze u-cycles, and find that any u-cycle contains one vertex of K (namely u itself), exactly two (non-adjacent) vertices of C, called the *C-vertices* of D, and at least one vertex of K'. Since, by definition, K' is connected and every vertex in C has a neighbor in K', graph G has at least one u-cycle for every pair v_i, v_j of C-vertices. However, only u-cycles of a special type specified below turn out to be useful. Since C contains at least four vertices and C-vertices v_i, v_j are not adjacent, the disjoint paths $P_1 = v_{i+1}\overrightarrow{C}v_{j-1}$ and $P_2 = v_{j+1}\overrightarrow{C}v_{i-1}$ are non-empty. If the vertices of P_1 and P_2 belong to a common component in $G[V \backslash V_D]$, then we say that D is a *correct u-cycle* of G. See Figure 8 for an example. Note that a correct u-cycle is not necessarily a W_4-contraction cycle of G. However, Lemma 4.10 explains that this fact does not matter: if G has a correct u-cycle, then G is W_4-contractible. Lemma 4.11 shows that correct u-cycles, in case they exist, can be found in polynomial time. So, our algorithm finishes as soon as it has found a correct u-cycle of G. If G does not have a correct u-cycle, our algorithm needs to perform a new step.

Step 6. Reduce the size of the vertex set

By now, we have an advanced state (C, K, K') such that K has a main vertex u with no correct u-cycles. Lemma 4.12 shows that in that case all W_4-contraction cycles of G (if they exist) do not contain a vertex of K. So far our algorithm has not reduced the size of G itself. However, as soon as we find out that two adjacent vertices u and v of G will never be on any W_4-contraction cycle C of G then both u and v will be in witness set X. (This is, of course, under the assumption that G is W_4-contractible.) Hence, we can contract the edge $[u, v]$ and apply our algorithm on the resulting graph. We show that we can indeed perform such contractions. First we consider the case, in which K contains more than one vertex. We use the information provided by Lemma 4.12. In Lemma 4.13, we show that our algorithm can remove all vertices of K not equal to u: the obtained graph is W_4-contractible if and only if G is W_4-contractible. So, we define $G := G[V \backslash (V_K \backslash \{u\})]$ with advanced state $(C, \{u\}, K')$ and without any correct u-cycles. Our purpose is to reduce the size of V even more. We first show in Lemma 4.14 that all W_4-contraction cycles of G (if they exist) contain at most two vertices of G such that if there are two such vertices then they are adjacent. Then in Lemma 4.15 we show that we may contract an edge $[u, v]$ for any $v \in C$ such that the resulting graph is W_4-contractible if and only if G is W_4-contractible. Our algorithm returns to Step 3 with a smaller 3-connected graph as input. Finally, in Theorem 9 and Corollary 10, we prove the correctness of the whole algorithm and show that it runs in polynomial time.

4.2 The algorithm itself

As explained in Section 4.1, our polynomial time algorithm for solving the W_4-Contractibility problem performs six steps. Here we describe them in more detail. We could prove some statements regarding W_k-contractibility for all $k \geq 4$. However, at some moment we need to restrict ourselves to $k = 4$: the computational complexity of the W_k-Contractibility problem for $k \geq 5$ is still open.

Step 1. Increase the connectivity as much as possible

Note that every wheel W_k is 2-connected. Hence, we can restrict ourselves to 2-connected input graphs due to Lemma 2.1. Since every wheel W_k is 3-connected, we can even prove a stronger result by using the following lemma. Recall that, for an induced subgraph F of a graph $G = (V, E)$ and two vertices $u, v \in V \backslash V_F$, we write $F + [u, v]$ to denote the graph obtained from $G[V_F \cup \{u, v\}]$ after adding the edge $[u, v]$.

Lemma 4.1 *Let $\{u, v\}$ be a 2-vertex cut of a 2-connected graph $G = (V, E)$, and let $k \geq 3$. Then G is W_k-contractible if and only if there exists a component $F \in G[V \backslash \{u, v\}]$ such that $F + [u, v]$ is W_k-contractible.*

Proof: Let $\mathcal{C}$ denote the set of all components in $G[V \backslash \{u, v\}]$. Suppose G is contractible to W_k. There are two cases to consider.

Case 1. Vertices u and v are in two different witness sets $W(i)$ and $W(j)$ of G. The witness sets of u and v must be adjacent because otherwise u or v is a cutvertex of G implying that G is not 2-connected. Since $\{u, v\}$ is a 2-vertex cut, the vertices in the other three witness sets must all belong to the same component F of $\mathcal{C}$. We remove all vertices in $W(i) \cup W(j)$ not in $V_F \cup \{u, v\}$. Note that the two resulting witness sets $W'(i)$ and $W'(j)$ are still connected. However, there might not be an edge between $W'(i)$ and $W'(j)$. By adding the edge $[u, v]$ we obtain a W_k-contractible graph.

Case 2. Vertices u and v are in the same witness set $W(i)$ of G. Since $\{u, v\}$ is a 2-vertex cut, the vertices in the other four witness sets must all belong to the same component F of $\mathcal{C}$. In this case, removing all vertices in $W(i)$ that are not in $V_F \cup \{u, v\}$ and adding the edge $[u, v]$ results in a W_k-contractible graph.

For the other direction of the proof, suppose there exists a component $F \in \mathcal{C}$ such that $F + [u, v]$ is W_k-contractible. Since we can contract $G[V \backslash V_F]$ to the graph $(\{u, v\}, \{[u, v]\})$, it is clear that G is W_k-contractible as well. □

Corollary 7 *Let $k \geq 3$. If the W_k-*Contractibility *problem is solvable in polynomial time for the class of 3-connected graphs, then the W_k-*Contractibility *problem is solvable in polynomial time.*

Proof: Let $k \geq 3$ and assume that the W_k-CONTRACTIBILITY is solvable in polynomial time for the class of 3-connected graphs. Let $G = (V, E)$ be a graph. The number of blocks of G is $O(|V|)$, and we can find all blocks of G in polynomial time. Since W_k is 3-connected, and consequently 2-connected, we can use Lemma 2.1. Hence it is sufficient to verify each block of G. Furthermore, we can find out in polynomial time whether a block B of G is 3-connected. If not, we can find a 2-vertex cut $\{u, v\}$ of B with set $\mathcal{C}$ of components in $B[V_B \backslash \{u, v\}]$ in polynomial time. Due to Lemma 4.1, we only have to check whether there exists some $F \in \mathcal{C}$ such that $F + [u, v]$ is W_k-contractible. If some $F + [u, v]$ with $F \in \mathcal{C}$ is not 3-connected, we repeat the procedure with respect to $F + [u, v]$. This way we find a total set of $O(|V|)$ 3-connected graphs, and each of these has at most $|V|$ vertices. We check each of them. □

Due to Corollary 7 we will only consider 3-connected input graphs from now on.

Step 2. Decrease the search space of possible W_4-witness structures

We show that a 3-connected graph G is W_k-contractible if and only if G has a W_k-contraction cycle, i.e., a chordless cycle $C = v_1 v_2 \ldots v_p v_1$ with $p \geq k$ such that $G[V \backslash V_C]$ is connected.

Lemma 4.2 *Let $k \geq 3$. A 3-connected graph G is W_k-contractible if and only if G has a W_k-contraction cycle.*

Proof: Suppose G has a W_k-contraction cycle $C = v_1 v_2 \ldots v_p v_1$. Since all vertices of a 3-connected graph G have at least degree three, any vertex on C has a neighbor in $G[V \backslash V_C]$. Hence, G is W_k-contractible with W_k-witness sets $Y_i = \{v_i\}$ for $1 \leq i \leq k-1$, $Y_k = \{v_k, \ldots, v_p\}$ and $X = V \backslash V_C$ (note that by definition $G[X]$ is connected).

To prove the reverse implication, let $G = (V, E)$ be 3-connected and W_k-contractible with witness sets X and $Y_1, \ldots, Y_k$. We say that a cycle C in $G[Y_1 \cup \ldots \cup Y_k]$ is a *cycle over* $Y_1, \ldots, Y_k$ if C contains some vertices $u_i, v_i \in Y_i$ (with possibly $u_i = v_i$) for $i = 1, \ldots, k$ such that $[u_{i-1}, v_i]$ for $i = 2, \ldots, k$ and $[u_k, v_1]$ are the only edges of C between two different witness sets.

We apply the following procedure to obtain witness sets X^* and $Y_1^*, \ldots, Y_k^*$ such that every neighbor of X^* is on every cycle over $Y_1^*, \ldots, Y_k^*$. As long as there exists a cycle C over $Y_1, \ldots, Y_k$ that does not contain a vertex $z \in N(X) \cap Y_i$ for some $1 \leq i \leq k$, move z to X. If z is a cutvertex of $G[Y_i]$, this operation results in a disconnected witness set Y_i'. Note that in this case all vertices of $V_C \cap Y_i$ are contained in one of the connected components, and we move all vertices that belong to the other components of $G[Y_i \backslash \{z\}]$ to X as well. We note that in each such iteration the size of X increases, and therefore this procedure terminates at some point.

Let $C = v_1 v_2 \ldots v_p v_1$ be an arbitrary cycle over $Y_1^*, \ldots Y_k^*$. From the above we find that C contains all $|N(X^*)| \geq k$ neighbors of X^*. We now prove that every vertex of C is a neighbor of X^*. Suppose $v_i \in V_C$ is not a neighbor of X^*. Let v_h be the vertex in $V_C \cap N(X^*)$ such that $v_i \overleftarrow{C} v_{h+1}$ does not contain any neighbors in X^*. Let v_j be the vertex in $V_C \cap N(X^*)$ such that $v_i \overrightarrow{C} v_{j-1}$ does not contain any neighbors in X^*. Because G is 3-connected, there exists a path $P = u_1 \ldots u_r$ in G from $v_i = u_1$ to a vertex u_r in X^*, such that v_h and v_j are not on P. Without loss of generality we may assume the following. Firstly, P is an induced path in G and P contains at most one edge between any two different witness sets Y_s^* and Y_t^* as otherwise we can easily redefine P. Secondly, v_i is the only vertex of P on $v_{h+1} \overrightarrow{C} v_{j-1}$ as otherwise we can redefine v_i to be the last such vertex along P (note that since $v_{h+1} \overrightarrow{C} v_{j-1}$ does not contain neighbors of X, this re-definition of v_i can be made).

Since $N(X^*) \subseteq V_C$, path P must contain a vertex of C. We let $u_q \in V_P \backslash \{v_i\}$ be the first vertex of P on C, i.e., vertex $u_q \in V_C$ and $u_i \notin V_C$ for $i = 2, \ldots, q-1$. By definition of P, vertex u_q is on the path $v_{j+1}\overrightarrow{C}v_{h-1}$. Then either $v_i\overrightarrow{C}u_q\overleftarrow{P}v_i$ is a cycle over $Y_1^*, \ldots Y_k^*$ that does not contain vertex $v_h \in N(X^*)$, or else $v_i\overrightarrow{P}u_q\overrightarrow{C}v_i$ is a cycle over $Y_1^*, \ldots, Y_k^*$ that does not contain $v_j \in N(X^*)$. In both cases we have constructed a cycle over $Y_1^*, \ldots, Y_k^*$ not containing all neighbors of X^*. This is again a contradiction. So the vertex set of any cycle over $Y_1^*, \ldots, Y_k^*$ consists of $N(X^*)$.

Let C be a cycle over $Y_1^*, \ldots, Y_k^*$. Then C is chordless. Otherwise we can easily construct a shorter cycle C' over $Y_1^*, \ldots, Y_k^*$, which would not contain all neighbors of X^*. We finish the proof by showing that each $G[Y_i^*]$ only contains vertices of C. Then, since C is chordless, each $G[Y_i]$ is a path as is desired.

Suppose a witness set, say Y_1^*, contains a vertex w_1 that is not on C. Since $V_C = N(X^*)$, vertex w_1 is not adjacent to X^*. We assume without loss of generality that w_1 is adjacent to a vertex v on $V_C \cap Y_1^*$. Because G is 3-connected, there exists a path $Q = w_1 \ldots w_r$ in G from w_1 to a vertex w_r in X^*, such that v is not on Q. Without loss of generality we assume that Q is an induced path in G and that Q contains at most one edge between any two different witness sets Y_s^* and Y_t^*.

Since $N(X^*) = V_C$, path Q must contain a vertex of C. We let $w_j \in V_Q$ be the first vertex of Q on C, i.e., vertex $w_j \in V_C$ and $w_i \notin V_C$ for $i = 1, \ldots, j-1$. Suppose w_j is not in Y_1. Then we may without loss of generality assume that w_j is in Y_2. Then the cycle $C' = w_1\overrightarrow{Q}w_j\overrightarrow{C}vw_1$ is a cycle over $Y_1^*, \ldots, Y_k^*$ containing a vertex (namely vertex w_1) not adjacent to X^*. This is a contradiction. Suppose w_j is in Y_1. Then either $w_1\overrightarrow{Q}w_j\overrightarrow{C}vw_1$ or else $w_1v\overrightarrow{C}w_j\overleftarrow{Q}w_1$ is a cycle over $Y_1^*, \ldots, Y_k^*$ containing vertex $w_1 \notin N(X^*)$. Hence, $G[Y_i^*]$ contains only vertices of C. □

Note that any W_k-contractible graph G with $k \geq 4$ is also W_j-contractible for $3 \leq j \leq k-1$. So W_k-contractibility is a necessary condition for W_{k+1}-contractibility. As a direct consequence of Lemma 4.2 we can even strengthen this observation.

Corollary 8 *Let G be a 3-connected graph that is W_k-contractible for some $k \geq 3$. If G does not contain a W_k-contraction cycle C on k vertices, then G is not W_{k+1}-contractible.*

In the rest of this section we will concentrate on the W_4-CONTRACTIBILITY problem. We will present a polynomial time algorithm that either finds a W_4-witness structure of a 3-connected graph G, or else outputs that G is not W_4-contractible.

Step 3. Find a chordless cycle on at least four vertices and do some easy checks

The next lemma provides us with a 'start state' for the algorithm.

Lemma 4.3 *Let G be a 3-connected graph. It is possible in polynomial time either to find a chordless cycle C on at least four vertices, or else to conclude that G is not W_4-contractible.*

Proof: For each three vertices u, v, w of G that induce a path uvw in G we act as follows. We remove v and $N(v) \backslash \{u, w\}$. Then we compute (in polynomial time) a shortest path P from w to u in the resulting graph G'. If such an induced path P of G' indeed exists, then $uvw\overrightarrow{P}u$ is a chordless cycle on at least four vertices in G. Otherwise we guess another triple of vertices of G and so on.

If G contains a chordless cycle $C = v_1v_2 \ldots v_pv_1$ on $p \geq 4$ vertices, then we will find a chordless cycle by performing the above procedure for the triple (v_1, v_2, v_3). Hence, G does not contain a chordless cycle on at least four vertices, if we have not found it after considering all $O(|V|^3)$ possible triples. In that case G is not W_4-contractible due to Lemma 4.2. □

Suppose for the 3-connected input graph G we found a chordless C cycle on at least four vertices. The following lemma describes the three easy situations in which we can immediately conclude that G is W_4-contractible.

Lemma 4.4 *Let $G = (V, E)$ be a 3-connected graph that contains a chordless cycle C on at least four vertices. Let $\mathcal{K}$ denote the set of components in $G[V \backslash V_C]$. Then G is W_4-contractible if one of the following cases is true:*

(i) there exists a vertex $v \in V_C$ that is not adjacent to any vertex in some $K \in \mathcal{K}$;

(ii) $|\mathcal{K}| = 1$ or $|\mathcal{K}| \geq 3$;

(iii) there exists a component $K \in \mathcal{K}$ that contains two vertices u_1 and u_2 that are both adjacent to all vertices of C.

Proof: We write $C = v_1 v_2 \dots v_p v_1$ with $p \geq 4$. We first prove that G is W_4-contractible if (i) holds. Suppose $v_i \in V_C$ is not adjacent to any vertex of $K \in \mathcal{K}$. Since G is 3-connected, V_K is adjacent to at least three vertices of C. Let v_h, v_j be two vertices in $N(V_K) \subseteq V_C$ such that K has a neighbor in neither $v_{h+1} \overrightarrow{C} v_i$ nor $v_i \overrightarrow{C} v_{j-1}$. Let $\mathcal{L} \subseteq \mathcal{K}$ be the set of all components of $G[V \backslash V_C]$ that only have neighbors on $v_h \overrightarrow{C} v_j$, i.e., $N(V_L) \subseteq V_{v_h \overrightarrow{C} v_j}$ for all $L \in \mathcal{L}$.

We will show that G is W_4-contractible with the following witness sets: $Y_1 = \{v_h\}, Y_2 = V_{v_{h+1} \overrightarrow{C} v_{j-1}} \cup \bigcup_{L \in \mathcal{L}} V_L, Y_3 = \{v_j\}, Y_4 = V_K$, and $X = V \backslash (Y_1 \cup Y_2 \cup Y_3 \cup Y_4)$. Obviously, the subgraphs $G[Y_1], G[Y_3]$ and $G[Y_4]$ are connected. Since G is 3-connected, any component in $\mathcal{L}$ has a neighbor on $v_{h+1} \overrightarrow{C} v_{j-1}$. Hence $G[Y_2]$ is connected. Since K does not belong to $\mathcal{L}$, its vertex set has a neighbor in $v_{j+1} \overrightarrow{C} v_{h-1}$. This implies that $v_{j+1} \overrightarrow{C} v_{h-1}$ is non-empty, and hence $[v_j, v_h]$ is not an edge of E_G. Then it is clear that $G[Y_1 \cup Y_2 \cup Y_3 \cup Y_4]$ is contractible to a chordless cycle on four vertices. By definition of $\mathcal{L}$, every component in $\mathcal{K} \backslash \mathcal{L}$ has a neighbor on $v_{j+1} \overrightarrow{C} v_{h-1}$. Then the induced subgraph $G[X]$ is connected. We are left to show that there is an edge between X and Y_i for all $1 \leq i \leq 4$.

Obviously, both Y_1 and Y_3 are adjacent to X. Recall that $Y_4 = V_K$ has a neighbor in X as well. Since G is 3-connected, there exists a path P from $v_i \in Y_2$ to a vertex u in K that contains neither v_h nor v_j. Hence, P must contain a vertex in $v_{j+1} \overrightarrow{C} v_{h-1} \subset G[X]$.

We now prove that G is W_4-contractible if (ii) holds. The case in which $\mathcal{K}$ contains only one component is trivial. Suppose $\mathcal{K}$ has at least three components. We assume that (i) is not valid, i.e., every vertex $v_i \in V_C$ is adjacent to the vertex set of every component in $\mathcal{K}$.

Let K, K' be two different components in $\mathcal{K}$. We claim that G is W_4-contractible with witness sets $Y_1 = \{v_1\}, Y_2 = V_K, Y_3 = \{v_3\}, Y_4 = V_{K'}$, and $X = V \backslash (Y_1 \cup Y_2 \cup Y_3 \cup Y_4)$. Obviously, $G[Y_1 \cup Y_2 \cup Y_3 \cup Y_4]$ is contractible to a chordless cycle on four vertices. So we are left to show that $G[X]$ is connected and that every Y_i has a neighbor in X.

Let L be a component in $\mathcal{K} \backslash \{K, K'\}$. Since (i) is not valid, all vertices of C have a neighbor in L. Hence there is a path (going through L) from v_2 to v_4 in $G[X]$. Since v_2 is adjacent to the vertex sets of *all* components of $\mathcal{K}$, the subgraph $G[X]$ is connected and every Y_i has a neighbor, namely v_2, in X.

Finally, we show that G is W_4-contractible if (iii) holds. We assume that (i) and (ii) are not valid for C. Since (ii) is not valid, $G[V \backslash V_C]$ consists of exactly two components K and K'. Suppose K contains two vertices u_1 and u_2 that are adjacent to all vertices of C. Since (i) is not valid, we may assume that every $v_i \in V_C$ has a neighbor in K'. Let w be a neighbor of v_1 in K' and let z be a neighbor of v_3 in K'. Since K' is connected, K' contains a path P from w to z. Suppose $t \in V_P \backslash \{w, z\}$ is a neighbor of v_1. Then we consider the subpath from t to z instead of P. If $t \in V_P \backslash \{w, z\}$ is a neighbor of v_3, then we consider the subpath from w to t instead of P. Hence, we may without loss of generality assume that the cycle $C' = u_1 v_1 w \overrightarrow{P} z v_3 u_1$ is a chordless cycle of G on at least four vertices.

If one of the vertices of C' is not adjacent to the vertex set of a component in $G[V \backslash V_{C'}]$, then G is W_4-contractible due to (i). Suppose every vertex of C' is adjacent to the vertex set of every component in $G[V \backslash V_{C'}]$. We will show that $G[V \backslash V_{C'}]$ is connected implying that G is W_4-contractible.

Because u_2 is adjacent to all vertices in C, the vertices $u_2, v_2, v_4, \ldots, v_p$ all belong to the same component L of $G[V \backslash V_{C'}]$. Suppose $s \in V \backslash V_{C'}$ is not in L. Then s is a vertex of K or of K'. Suppose s is in K. Let L' be the component of $G[V \backslash V_{C'}]$ that contains s. Then, since $V_C \subseteq V_L \cup V_{C'}$, we conclude that L' is contained in K. If w does not have a neighbor in L', then G is W_4-contractible due to (i). Let s' be a neighbor of w in $L' \subset G[V \backslash V_{C'}]$. Then $s' \notin V_K$ and this contradicts the fact that $L' \subseteq K$. If s is a vertex of K', we can apply a similar argument to obtain a contradiction. Hence, $G[V \backslash V_{C'}]$ is connected implying that G is W_4-contractible. □

Let G be a 3-connected graph with a chordless cycle C on at least four vertices. Recall that in case $G[V \backslash C]$ consists of exactly two components K and K' then G induces a basic state (C, K, K') of G, so

(1) every vertex in C has a neighbor in both K and K', and

(2) both K and K' contain at most one vertex that is adjacent to all vertices of C,

By checking the three conditions (i)-(iii) of Lemma 4.4, it can be verified in polynomial time whether a chordless cycle C on at least four vertices induces a basic state of a 3-connected graph G or not. As a direct consequence of (the proof of) Lemma 4.4, we find the following.

Lemma 4.5 *Let G be a 3-connected graph with chordless cycle C on at least four vertices. If C does not induce a basic state, then a W_4-witness structure of G can be found in polynomial time.*

Step 4. Decrease the size of one of the two remaining components

Our next goal is to decrease the size of component K of a basic state (C, K, K') of a 3-connected input graph G as much as possible. For this purpose we need the following lemma.

Lemma 4.6 *Let (C, K, K') be a basic state of a 3-connected graph G, such that $G[V_C \cup V_K]$ contains a chordless cycle D that has at least four vertices and that has at least one vertex in K. Then G is W_4-contractible or D induces a basic state (D, L, L') of G with $L \subset K$.*

Proof: If D does not induce a basic state of G, then G is W_4-contractible due to Lemma 4.5. Suppose D induces a basic state (D, L, L') of G. Since (C, K, K') is a basic state of G, all vertices of C have a neighbor in K'. Then we can choose L' to be the component that contains all vertices in $V_{K'} \cup V_C \backslash V_D$ (plus possibly some vertices of K). This means that all vertices in L are in $V_K \backslash V_D$. Since D contains at least one vertex of K, we find that $V_L \subseteq V_K \backslash V_D$ is a proper subset of V_K. □

Let (C, K, K') be a basic state of a 3-connected graph $G = (V, E)$. Recall that (C, K, K') is an advanced state if $G[V_C \cup V_K]$ does not contain a chordless cycle D on at least four vertices with at least one vertex in K. Note that if (C, K, K') is an advanced state of G, then any W_4-contraction cycle D of a W_4-contractible, 3-connected graph G must contain at least one vertex of K'. We now show it is possible to find an advanced state of G in polynomial time.

Lemma 4.7 *Let (C, K, K') be a basic state of a 3-connected graph G. It is possible in polynomial time either to find a W_4-witness structure of G, or else to find an advanced state (D, L, L') of G.*

Proof: Let (C, K, K') be a basic state of a 3-connected graph $G = (V, E)$. We apply the following procedure. We consider all $O(|V|^3)$ triples u, v, w in $V_C \cup V_K$ that induce a path uvw in $G[V_C \cup V_K]$ with $\{u, v, w\} \cap V_K \neq \emptyset$, until we find such a triple u, v, w for which there exists a path P from w to u in $G[(V_C \cup V_K) \backslash (\{v\} \cup N(v))]$. If we do not find such a path P for any triple, then (C, K, K') is an advanced state. Otherwise, in case we do find such a path P, the cycle $D = uvw\overrightarrow{P}u$ is a chordless cycle of $G[V_C \cup V_K]$ that has at least four vertices and that contains at least one vertex in K. By Lemma 4.6, graph G is either W_4-contractible or D induces a basic state (D, L, L') of G with $L \subset K$. Note that we can check in polynomial time whether D induces a basic state of G. If D does not induce a basic state, then we can find W_4-witness sets of G in polynomial time according to Lemma 4.5. If D induces a basic state (D, L, L') of G with $L \subset K$, then we apply the whole procedure again, but this time with respect to D. After considering a sequence of $O(|V|)$ chordless cycles D_i that induce basic states (D_i, L_i, L_i') of G, in which the size of L_i is strictly decreasing, we have either found W_4-witness sets of G, or else an advanced state of G. □

Step 5. Detect a vertex adjacent to all vertices in the cycle and do an easy check

Let (C, K, K') be an advanced state of a 3-connected graph so it is also a basic state. By definition, K contains at most one vertex adjacent to all vertices of C. Recall that we call such a vertex u the main vertex of K. In order to continue we need some information on the structure of an advanced state of a 3-connected graph.

Lemma 4.8 *Let (C, K, K') be an advanced state of a 3-connected graph G. Then the following holds:*

(i) only a main vertex in K is adjacent to two non-adjacent vertices in C, and

(ii) K contains one (and exactly one) main vertex.

Proof: We write $C = v_1 v_2 \dots v_p v_1$ for some $p \geq 4$. We will first show that (i) holds. Suppose s is a vertex of K that is not a main vertex but that is adjacent to two non-adjacent vertices $v_h, v_j \in V_C$. Assume that v_h and v_j are chosen in such a way that s is not adjacent to any vertex on $v_{h+1}\overrightarrow{C}v_{j-1}$. Then we have found a chordless cycle $D = sv_h\overrightarrow{C}v_j s$ of $G[V_C \cup V_K]$ that has at least four vertices and that contains a vertex of K. This contradicts our assumption that (C, K, K') is an advanced state of G. We will now show that (ii) holds.

We first prove that all adjacent vertices of C have a common neighbor in K. Suppose C contains two adjacent vertices, say v_1, v_2, that do not have a common neighbor in K. Since (C, K, K') is an advanced state of G, it is also a basic state of G, and therefore vertex v_1 has a neighbor $u \in V_K$ and vertex v_2 has a neighbor $w \in V_K$. Since we assume that v_1 and v_2 do not have a common neighbor in K, vertices u and w are two different vertices in K. Since K is connected, there exists a path from u to w in K. Let P be a shortest path from u to w in K. If P contains a vertex $z \neq u$ adjacent to v_1, we consider the subpath from z to w instead of P. If P contains a vertex $z \neq w$ adjacent to v_2, we consider the subpath from u to z instead of P. So we may assume that u and w are the only vertices of P adjacent to v_1 and v_2. Then $D = v_1 u\overrightarrow{P}wv_2v_1$ is a chordless cycle of $G[V_C \cup V_K]$ that has at least four vertices and that has a vertex in K. This contradicts our assumption that (C, K, K') is an advanced state of G. So we have found that every pair of adjacent vertices on C has a common neighbor in K.

Suppose K does not contain a main vertex. Due to the above, v_1 and v_2 have a common neighbor t in K and v_2, v_3 have a common neighbor t' in K. Due to (i), vertices t and t' are different vertices. Let P be a shortest path from t to t' in component K. If P contains a common neighbor s of v_1 and v_2 not equal to t, then we consider the subpath from s to t' instead of P. So we may assume that t is the only common neighbor of v_1 and v_2 on P. By using the same argument we may also assume that t' is the only common neighbor of v_2 and v_3 on P.

Suppose all vertices of P are adjacent to v_2. Due to (i), all vertices of P do not have a neighbor in $v_4\overrightarrow{C}v_p$. Then the cycle $D = v_1t\overrightarrow{P}t'v_3\overrightarrow{C}v_1$ is a chordless cycle of $G[V_K \cup V_C]$ that has at least four vertices and that has at least two vertices in K. This contradicts our assumption that (C, K, K') is an advanced state.

Suppose P contains a vertex u not adjacent to v_2. Let s and s' be vertices of P such that the path $s\overrightarrow{P}s'$ contains u and does not contain any neighbors of v_2 other than s and s'. Then $D = v_2s\overrightarrow{P}s'v_2$ is a chordless cycle of $G[V_C \cup V_K]$ that has at least four vertices and that has at least three vertices in K. This contradicts our assumption that (C, K, K') is an advanced state. We conclude that K has a main vertex (which is the only one by definition of a basic state). $\square$

Let (C, K, K') be an advanced state of a 3-connected graph G. Recall that we denote its main vertex, which exists due to Lemma 4.8, by u, and that we call a chordless cycle D of G on at least four vertices that contains u a u-cycle of G. In the lemma below we describe the structure of u-cycles.

Lemma 4.9 *Let (C, K, K') be an advanced state of a 3-connected graph G with main vertex u. Let D be a u-cycle of G. Then D contains at least one vertex of K', exactly two vertices v_i, v_j of C, which are not adjacent, and exactly one vertex of K (namely the main vertex u).*

Proof: We write $C = v_1v_2 \dots v_pv_1$ for some $p \geq 4$. Let D be a u-cycle of G. Since (C, K, K') is an advanced state of G, cycle D must contain at least one vertex of K'. Since u does not have a neighbor in K', the u-cycle D contains at least two vertices v_i, v_j of C. Because u is adjacent to all vertices of C and D is chordless, we find the following: First v_i and v_j are not adjacent. Secondly, v_i and v_j are the only vertices of D that are on C. Thirdly, D does not contain any other vertices of K besides u. $\square$

Let (C, K, K') be an advanced state of a 3-connected graph G with main vertex u. Recall that the vertices of a u-cycle D of G that are on C are called C-vertices of D. By Lemma 4.9, D has exactly two C-vertices. Since K' is connected and every vertex in C has a neighbor in K', graph G has at least one u-cycle for every pair v_i, v_j of C-vertices. However, as we explained in Section 4.1, only u-cycles that are correct turn out to be useful. Recall that a u-cycle D with C-vertices v_i, v_j is called correct, if the (disjoint and nonempty) paths $P_1 = v_{i+1}\overrightarrow{C}v_{j-1}$ and $P_2 = v_{j+1}\overrightarrow{C}v_{i-1}$ belong to a common component in $G[V \backslash V_D]$. A correct u-cycle is not necessarily a W_4-contraction cycle of G, however this does not matter as we see in the following lemma.

Lemma 4.10 *Let (C, K, K') be an advanced state of a 3-connected graph G with main vertex u. If G has a correct u-cycle, then G is W_4-contractible.*

Proof: We write $C = v_1v_2 \dots v_pv_1$ for some $p \geq 4$. Let D be a correct u-cycle with C-vertices v_i and v_j. We show that D does not induce a basic state of G. Then, by Lemma 4.5, G is W_4-contractible.

Suppose D induces a basic state (D, L, L') of G. We assume without loss of generality that L contains the vertices on $v_{i+1}\overrightarrow{C}v_{j-1}$. Since D is a correct u-cycle, also all vertices of $v_{j+1}\overrightarrow{C}v_{i-1}$ belong to L. Since L' is connected and any vertex of C is in $V_D \cup V_L$, either L' is a subgraph of K or L' is a subgraph of K'. Suppose L' is a subgraph of K. By Lemma 4.9, u-cycle D contains a vertex z of K'. Because (D, L, L') is a basic state, each vertex in D, and hence also z has a neighbor in $L' \subset K$. This is not possible though, because z belongs to K', and K' is not adjacent to K by definition of (C, K, K'). Suppose L' is a subgraph of K'. Since (D, L, L') is a basic state, vertex $u \in V_K \cap V_D$ must have a neighbor in $L' \subset K'$. Again this is not possible. $\square$

We will now show that correct u-cycles, in case they exist, can be found in polynomial time.

Lemma 4.11 *Let (C, K, K') be an advanced state of a 3-connected graph G with main vertex u. It is possible in polynomial time either to find a correct u-cycle of G, or else to conclude that G does not have a correct u-cycle.*

Proof: We write $C = v_1v_2 \ldots v_pv_1$ for some $p \geq 4$. Due to Lemma 4.9, every u-cycle contains at least one vertex of K', exactly two C-vertices, and no other vertices of K besides u. This means we can apply the following procedure.

Let v_i and v_j be two non-adjacent vertices on C. Since (C, K, K') is an advanced state, every vertex on C is adjacent to at least one vertex in K'. Let w be a neighbor of v_i in K', and let z be a neighbor of v_j in K'. We remove u, v_i, v_j, and perform a series of edge contractions that contract $v_{i+1}\overrightarrow{C}v_{j-1}$ into a new vertex x_1 and $v_{j+1}\overrightarrow{C}v_{i-1}$ into a new vertex x_2. We call the resulting graph G^*. Suppose G^* contains an induced path P from x_1 to x_2 and an induced path P' from w to z, such that P and P' are vertex-disjoint, i.e., $V_P \cap V_{P'} = \emptyset$. Then P' is a path in K', and the cycle $uv_iw\overrightarrow{P'}zv_ju$ is a correct u-cycle of G. Shiloach [5] gives an algorithm that finds such vertex-disjoint paths P and P' in G^*, if they exist, in polynomial time.

If the choice of neighbors w, z does not result in a correct u-cycle, then we try other neighbors of v_i and v_j in K'. If we still do not obtain a correct u-cycle, then we choose another pair of non-adjacent vertices on C, and repeat the procedure. Since the total number of guesses is $O(|V|^4)$ and each guess can be checked in polynomial time, we establish the claim of the lemma. $\square$

Due to Lemma 4.11 our algorithm finishes as soon as it has found (in polynomial time) a correct u-cycle of G. In the next step we describe how to proceed if G does not have any correct u-cycle.

Step 6. Reduce the size of the vertex set

Suppose we have an advanced state (C, K, K') of a 3-connected graph G with main vertex u of K and with no correct u-cycles. In order to proceed we first have to prove the following structural result.

Lemma 4.12 *Let (C, K, K') be an advanced state of a 3-connected graph G with main vertex u. If G does not have a correct u-cycle, then any W_4-contraction cycle D of G does not contain any vertices of K.*

Proof: We write $C = v_1v_2 \ldots v_pv_1$ for some $p \geq 4$. Suppose G does not have a correct u-cycle. Suppose $D = v'_1v'_2 \ldots v'_qv'_1$ with $q \geq 4$ is a W_4-contraction cycle of G that contains a vertex, say v'_1, of K.

Suppose D is a u-cycle. By Lemma 4.9, D contains (exactly) two C-vertices v_i and v_j. Since D is a W_4-contraction cycle, the subgraph $G[V \backslash V_D]$ is connected. Then the paths $v_{i+1}\overrightarrow{C}v_{j-1}$ and $v_{j+1}\overrightarrow{C}v_{i-1}$ are connected in $G[V \backslash V_D]$. Then D would be a correct u-cycle of G. Hence, u is not on D.

Since (C, K, K') is an advanced state of G, by definition, D contains a vertex of K'. Since K and K' are not adjacent and $v'_1 \in V_D \cap V_K$, cycle D must contain at least two different vertices of C. Let v'_i be the vertex of $V_D \cap V_C$ such that $P_1 = v'_1\overrightarrow{D}v'_{i-1}$ only contains vertices of K. Let

v'_j be the vertex of $V_D \cap V_C$ such that $P_2 = v'_{j+1}\overrightarrow{D}v'_1$ only contains vertices of K. If $v'_i = v'_j$, then $D = v'_1\overrightarrow{P_1}v'_{i-1}v'_iv'_{i+1}\overrightarrow{P_2}v'_1$, and if $[v'_i, v'_j] \in E_G$ then $D = v'_1\overrightarrow{P_1}v'_{i-1}v'_iv'_jv'_{j+1}\overrightarrow{P_2}v'_1$. However, since D contains a vertex of K', these two cases do not occur. So v'_i and v'_j are two different, non-adjacent vertices of C.

We consider the cycle $D_1 = v'_1\overrightarrow{D}v'_i\overrightarrow{C}v'_j\overrightarrow{D}v'_1$. Since v'_i and v'_j are not adjacent, D_1 is a cycle in $G[V_C \cup V_K]$ on at least four vertices. Let v be the successor of v'_i on D_1, and let v^* be the precessor of v'_j on D_1. Note that both v and v^* are on C. Since (C, K, K') is an advanced state of G, cycle $D_1 \subset G[V_C \cup V_K]$ is not a chordless cycle of G. Since both D and C are chordless cycles of G, we then find that there exists an edge between a vertex on $v\overrightarrow{D_1}v^* = v\overrightarrow{C}v^*$ and a vertex on $v'_{j+1}\overrightarrow{D_1}v'_{i-1} = v'_{j+1}\overrightarrow{D}v'_{i-1}$.

Let $v_h \in V_C$ be the first vertex on the path $v\overrightarrow{C}v^*$ that has a neighbor v'_r in $v'_{j+1}\overrightarrow{D}v'_{i-1}$, and assume that v'_r is the first neighbor of v_h along $v'_{j+1}\overrightarrow{D}v'_{i-1}$, so $D' = v'_i\overrightarrow{C}v_hv'_r\overrightarrow{D}v'_i$ is a chordless cycle of G. This means that D' is a chordless cycle of $G[V_C \cup V_K]$. Since (C, K, K') is an advanced stated of G, the chordless cycle D' must contain three vertices. Hence, $v_h = v$, and $v'_r = v'_{i-1}$ implying that v'_{i-1} is adjacent to the successor of v'_i on C. By using the same arguments with respect to $D_2 = v'_1\overrightarrow{D}v'_i\overleftarrow{C}v'_j\overrightarrow{D}v'_1$, we deduce that v'_{i-1} is also adjacent to the precessor of v'_i on C. Recall that u is not on D, so $v'_{i-1} \neq u$. By Lemma 4.8, vertex $v'_{i-1} \in V_K$ can not be adjacent to two non-adjacent vertices on C. Hence, the lemma has been proven. □

The following lemmas show that we may reduce the size of G. First we consider the case, in which K contains more than one vertex.

Lemma 4.13 *Let (C, K, K') be an advanced state of a 3-connected graph $G = (V, E)$ with main vertex u, such that G does not have a correct u-cycle. Then G is W_4-contractible if and only if $G[V_C \cup \{u\} \cup V_{K'}]$ is W_4-contractible.*

Proof: If $V_K = \{u\}$, the lemma is true. Suppose $|V_K| \geq 2$. Define $G' = G[V_C \cup \{u\} \cup V_{K'}]$. Suppose G' is W_4-contractible. Since G' can be obtained after a sequence of $|V_K| - 1$ edge contractions, graph G is W_4-contractible.

For the other direction of the proof, suppose G is W_4-contractible. Then G has a W_4-contraction cycle C^* due to Lemma 4.2. By Lemma 4.12, all vertices of K are in $V \backslash V_{C^*}$. As C^* is a W_4-contraction cycle, the graph $G[V \backslash V_{C^*}]$ is connected. Since $u \in V_K$ is adjacent to all vertices on C, we can remove all vertices of $V_K \backslash \{u\}$ from G without breaking the connectivity of $G[V \backslash V_{C^*}]$. Every vertex in V_{C^*} that is on C is adjacent to vertex u. Every vertex in V_{C^*} that is not on C is in K' and, consequently, not adjacent to V_K. Then C^* is a W_4-contraction cycle of G' (note that G' is 3-connected). By Lemma 4.2, graph G' is W_4-contractible. □

By Lemma 4.13 we may assume that we have a graph G in an advanced state $(C, \{u\}, K')$ such that G does not contain a correct u-cycle. We need one last structural result.

Lemma 4.14 *Let D be a W_4-contraction cycle of a 3-connected graph G that is in advanced state $(C, \{u\}, K')$ without any correct u-cycles. Then D contains at most two vertices of C. If D contains exactly two vertices of C, then these two vertices are adjacent.*

Proof: We write $C = v_1v_2 \dots v_pv_1$ for some $p \geq 4$. We use a proof by contradiction. Suppose $D = v'_1v'_2 \dots v'_qv'_1$ with $q \geq 4$ is a W_4-contraction cycle of G that does not satisfy the claim of the lemma. Then D either contains at least three vertices of C, or else exactly two vertices of C, but those two vertices are not adjacent.

Claim 1. u is not on D.

This claim immediately follows from Lemma 4.12.

Claim 2. D contains two non-adjacent vertices v'_h, v'_j such that $v'_{h+1}\overrightarrow{D}v'_{j-1}$ only contains vertices of K'.

We prove Claim 2 as follows. By Claim 1, u is not on D. Then Claim 2 is immediately clear if D contains exactly two vertices of C (since those two vertices are non-adjacent). Suppose D contains at least three vertices of C. Since D is chordless and has at least four vertices, we find that D contains a pair of non-adjacent vertices v'_h, v'_j of C. Since $D \neq C$, we may without loss of generality assume that $v'_{h+1}\overrightarrow{D}v'_{j-1}$ contains a vertex v'_i in $V_D \backslash V_C$. But then we may even assume that $v'_{h+1}\overrightarrow{D}v'_{j-1}$ only contains vertices from K'. If not, we can replace the pair v'_h, v'_j by another pair of non-adjacent vertices v'_k, v'_ℓ of C that are on $v'_h\overrightarrow{D}v'_j$, such that $v_{k'+1}\overrightarrow{D}v'_i\overrightarrow{D}v_{\ell'-1}$ only contains vertices of K'. This finishes the proof of Claim 2.

Claim 3. The path $v'_{j+1}\overrightarrow{D}v'_{h-1}$ has a neighbor in $V\backslash V_D$ not equal to u.

We prove Claim 3 as follows. First recall that, since G is 3-connected, all vertices on D have a neighbor in $V\backslash V_D$. We consider v'_{j+1}. There are two cases: either v'_{j+1} belongs to K', or v'_{j+1} belongs to C. Suppose v'_{j+1} is a vertex of K'. Then any neighbor of v'_{j+1} in $V\backslash V_D$ is not equal to u, because u is not adjacent to K'. Suppose v'_{j+1} is a vertex of C. Since $(C, \{u\}, K')$ is an advanced state, every vertex of C has a neighbor in K'. If v'_{j+2} is on C (this may happen for example in the case that $v'_{j+2} = v'_h$) then since D is chordless, such a neighbor of v'_{j+1} must be in $V\backslash V_D$. Suppose v'_{j+2} is in K'. Note that this implies that v'_{j+2} is not equal to v'_h, so v'_{j+2} is on $v'_{j+1}\overrightarrow{D}v'_{h-1}$. Hence we can use the same argument as before: any neighbor of v'_{j+2} in $V\backslash V_D$ is not equal to u, because u is not adjacent to K'. This completes the proof of Claim 3.

We define the u-cycle $D' = uv'_h\overrightarrow{D}v'_ju$. Since G does not have a correct u-cycle, $G[V\backslash V_{D'}]$ is not connected. Otherwise D' would be a W_4-contraction cycle, and therefore a correct u-cycle of G by the same arguments as in the proof of Claim 1. We denote the set of components of $V\backslash V_{D'}$ by $\mathcal{C}$, so $|\mathcal{C}| \geq 2$. Let $L \in \mathcal{C}$ contain $v'_{j+1}\overrightarrow{D}v'_{h-1}$.

Claim 4. Each component in $\mathcal{C}$ contains a vertex of $V_C\backslash V_D$.

We prove Claim 4 as follows. First consider the components in $\mathcal{C}$ not equal to L. Let L' be a component in $\mathcal{C}\backslash L$. Then all vertices of L' belong to $V\backslash V_D$. Since $G[V\backslash V_D]$ is connected and contains u, L' contains a vertex adjacent to u. Since u is not adjacent to K', such a vertex must be in C. We now consider L. By Claim 2, $v'_{j+1}\overrightarrow{D}v'_{h-1}$ has a neighbor in $V\backslash(V_D \cup \{u\})$. Hence L is not equal to $v'_{j+1}\overrightarrow{D}v'_{h-1}$ and contains at least one vertex z of $V\backslash V_D$. Since $G[V\backslash V_D]$ is connected and contains u, $G[V\backslash V_D]$ contains a path P_{zu} from z to u. So all vertices of P_{zu} belong to $V_L\backslash V_D$. Since u is not adjacent to K', P_{zu} contains a vertex of C. This finishes the proof of Claim 4.

Let $P \subset C$ denote the path from the successor of v'_h in C to the precessor of v'_j in C. Let $P' \subset C$ denote the path from the successor of v'_j in C to the precessor of v'_h in C. We observe that all vertices of P and P' are not on D', i.e., $(V\backslash V_{D'}) \cap V_C = V_P \cup V_{P'}$. By Claim 4, we may assume without loss of generality that L contains a vertex of P, and consequently all vertices of P. Recall that $|\mathcal{C}| \geq 2$. Let $L' \neq L$ be another component of $\mathcal{C}$. By Claim 4, L' contains a vertex of $V_C\backslash V_D$. Hence L' contains a vertex of P', and consequently all vertices of P'. We observe that, again due to Claim 4,

$$\mathcal{C} = \{L, L'\}.$$

Claim 5. Graph G contains a path Q from a vertex $v_r \in V_P \backslash V_D = (V_L \cap V_C) \backslash V_D$ to a vertex v'_a on $v'_{h+1}\overrightarrow{D}v'_{j-1}$ such that all vertices of $V_Q \backslash \{v'_a, v_r\}$ belong to $(V_L \cap V_{K'}) \backslash V_D$.

We prove Claim 5 as follows. By Claim 4, we can choose a vertex $v_r \in V_P \backslash V_D$. By definition of $(C, \{u\}, K')$, all vertices of C are adjacent to K' and K' is connected. So there exists a path Q' from $v_r \in V_C$ to a vertex v'_a in $v'_{h+1}\overrightarrow{D}v'_{j-1}$ such that $V_{Q'} \backslash \{v_r\}$ belongs to K'. We may without loss of generality assume that v'_a is the only vertex of $v'_{h+1}\overrightarrow{D}v'_{j-1}$ on Q': if not we just take a subpath of Q' from v_r to the first vertex of $v'_{h+1}\overrightarrow{D}v'_{j-1}$ on Q'.

If v'_a is the only vertex of Q' that is on D, then we take $Q = Q'$ and we have proven Claim 5. Suppose Q contains more vertices of D. Let $v'_d \neq v'_a$ be another vertex of Q' that is on D. By construction of Q', v'_d is not on $v'_{h+1}\overrightarrow{D}v'_{j-1}$. So v'_d belongs to $v'_{j+1}\overrightarrow{D}v'_{h-1}$. Then, since D is chordless, there exists a vertex $t \in V_L \backslash V_D$ such that we can write $Q' = v_r\overrightarrow{Q'}v'_d\overrightarrow{Q'}t\overrightarrow{Q'}v'_a$. We may without loss of generality assume that $t\overrightarrow{Q'}v'_a$ does not contain any vertex v'_e in $v'_{j+1}\overrightarrow{Q'}v'_{h-1}$. Otherwise we could just replace v'_d by v'_e (and then find a vertex $t' \in V_L \backslash V_D$ instead of t and so on).

Since $G[V \backslash V_D]$ is connected, t and u are in the same component of $G[V \backslash V_D]$. Hence there exists a path Q^* from u to t in $G[V \backslash V_D]$. Note that all vertices of Q^* except u belong to L, since t belongs to L. Let v^*_r be a neighbor of u on Q^*. Since u only has neighbors on C, v^*_r belongs to C. We choose v^*_r instead of v_r. Then we let $Q = v^*_r\overrightarrow{Q^*}t\overrightarrow{Q'}v'_a$ (possibly, after deleting some cycles that may exist if $v^*_r\overrightarrow{Q^*}t$ and $t\overrightarrow{Q'}v'_a$ overlap in other vertices than t only). This finishes the proof of Claim 5.

Claim 6. Graph G contains a path R from a vertex $v_s \in V_{P'} = V_{L'} \cap V_C$ to a vertex v'_b on $v'_{h+1}\overrightarrow{D}v'_{j-1}$ such that all vertices of $V_R \backslash \{v'_b, v_s\}$ belong to $(V_{L'} \cap V_{K'}) \backslash V_D$.

We prove Claim 6 as follows. Let v_s be a vertex of P'. By definition of $(C, \{u\}, K')$, all vertices of C are adjacent to K' and K' is connected. So there exists a path R' from $v_s \in V_C$ to a vertex v'_b in $v'_{h+1}\overrightarrow{D}v'_{j-1}$ such that $V_{R'} \backslash \{v_s\}$ belongs to K'. We may without loss of generality assume that v'_b is the only vertex of $v'_{h+1}\overrightarrow{D}v'_{j-1}$ on R': if not we just take a subpath of R' from v_s to the first vertex of $v'_{h+1}\overrightarrow{D}v'_{j-1}$. Since $v'_{j+1}\overrightarrow{D}v'_{h-1}$ belongs to L, we then find that v'_b is the only vertex of Q' that is on D. Hence we can take $R = R'$. This finishes the proof of Claim 6.

We note that Q and R may only have v'_a as a common vertex (in that case $v'_a = v'_b$) because Q without v'_a and R without v'_b belong to L and L' respectively.

Consider the u-cycle D^* formed by u, the vertices from Q, R and the path between v'_a and v'_b on $v'_{h+1}\overrightarrow{D}v'_{j-1}$. Since v_r is on P and v_s is on P', the vertices v'_h and v'_j are in two different segments of C after removing v_r and v_s. However, recall that the vertices of Q belong to $(V \backslash V_D) \cup \{v'_a\}$ and that the vertices of R belongs to $(V \backslash V_D) \cup \{v'_b\}$. Hence, due to this choice of Q and R, none of the vertices of $v'_j\overrightarrow{D}v'_h$ belong to D^*. In other words, $G[V \backslash V_{D^*}]$ contains the path $v'_j\overrightarrow{D}v'_h$. This means that D^* is a correct u-cycle. By this contradiction we have finished the proof of Lemma 4.14. □

We now show the final step of our algorithm (and then we can state our main theorem). Recall that $G \backslash e$ denotes the graph obtained from $G = (V, E)$ after contracting the edge $e \in E$.

Lemma 4.15 *Let $(C, \{u\}, K')$ be an advanced state of a 3-connected graph G that does not have a correct u-cycle. Let v be a vertex of C. Then G is W_4-contractible if and only if $G \backslash [u, v]$ is W_4-contractible.*

Proof: Let v be a vertex on C. Since $G \backslash [u, v]$ is obtained after an edge contraction, G is W_4-contractible if $G \backslash [u, v]$ is W_4-contractible.

For the other direction of the proof, suppose G is W_4-contractible. Then G has a W_4-contraction cycle C^* due to Lemma 4.2. By Lemma 4.12, vertex u is in $V \backslash V_{C^*}$. Suppose v is in $V \backslash V_{C^*}$. Then C^* is a W_4-contraction cycle of $G \backslash [u, v]$. Hence, by Lemma 4.2, the graph $G \backslash [u, v]$ is W_4-contractible. Suppose v is on C^*. If v is the only vertex of C that is on C^*, then $G[V \backslash (V_{C^*} \cup \{u\}]$ is connected. Suppose v is not the only vertex of C that is on C^*. Then, due to Lemma 4.14, cycle C^* contains one other vertex v' of C and $[v, v']$ is an edge in G. Again, we find that $G[V \backslash (V_{C^*} \cup \{u\}]$ is connected. Recall that u is only adjacent to vertices in C. Hence, in both cases, we can move u to the witness set that contains v, i.e., contract $[u, v]$, such that the resulting graph $G \backslash [u, v]$ is still W_4-contractible. □

Theorem 9 *Let G be a 3-connected graph. It is possible in polynomial time either to find a W_4-witness structure of G or else to conclude that G is not W_4-contractible.*

Proof: Let G be a 3-connected graph. According to Lemma 4.2 finding a W_4-witness structure of G is equivalent to finding a chordless cycle $C \subset G$ on at least four vertices such that $G[V \backslash V_C]$ is connected. By Lemma 4.3 we deduce that either G is not W_4-contractible, or else we find a chordless cycle $C \subseteq G$ on at least four vertices. Due to the same lemma, we can do this in polynomial time. We can check in polynomial time whether C induces a basic state of G. If C does not induce a basic state then we find a W_4-witness structure of G in polynomial time by using Lemma 4.5. In the other case we apply Lemma 4.7 to find in polynomial time either a W_4-witness structure of G, or else an advanced state (D, K, K') of G.

Suppose we find an advanced state (D, K, K') of G. By Lemma 4.11 we can find out in polynomial time whether G has a correct u-cycle for the main vertex u of K. Suppose G has a correct u-cycle. By the proof of Lemma 4.10, this u-cycle does not induce a basic state of G. By Lemma 4.5 we find a W_4-witness structure of G in polynomial time.

Suppose G does not have a correct u-cycle. By Lemma 4.13, graph G is W_4-contractible if and only if $G' = G[V_D \cup \{u\} \cup V_{K'}]$ is W_4-contractible. It is clear that G' is 3-connected. If it turns out that G' is W_4-contractible, then we obtain a W_4-witness structure of G from W_4-witness sets of G' by adding the vertices of $V_K \backslash \{u\}$ to the W_4-witness set of G' that contains u.

If G' does not have a correct u-cycle, then we choose an arbitrary vertex on D and contract the edge $[u, v]$. By Lemma 4.15, graph G' is W_4-contractible if and only if the smaller graph $G' \backslash [u, v]$ is W_4-contractible. For given W_4-witness sets of $G' \backslash [u, v]$ it is easy to construct W_4-witness sets of G': we remove the new vertex u^* in $G' \backslash [u, v]$, obtained after contracting $[u, v]$, from its W_4-witness set, and replace it by u, v.

Since $(C, \{u\}, K)$ is an advanced state of G', every vertex in C has a neighbor in K'. Then, $G'[V_{G'} \backslash \{u\}]$ contains three disjoint paths from v to its precessor on C and three disjoint paths to its successor on C. Since any other three disjoint paths between any two vertices $s \neq u$ and $t \neq u$ in G' can easily be replaced by three disjoint paths from s to t in $G' \backslash [u, v]$, graph $G' \backslash [u, v]$ is 3-connected. Hence, we may apply our algorithm for 3-connected graphs on the smaller graph $G' \backslash [u, v]$. After at most $|V|$ calls of this algorithm, we either found a W_4-witness structure of G, or else concluded that G is not W_4-contractible. □

Due to Corollary 7 and Theorem 9 we immediately obtain the main result of this section.

Corollary 10 *The W_4-*Contractibility *problem is solvable in polynomial time. Moreover, if a graph G is W_4-contractible, a W_4-witness structure of G can be found in polynomial time.*

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