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Matching games: The least core and the nucleolus

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MATCHING GAMES: THE LEAST CORE AND THE NUCLEOLUS

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ABSTRACT. A matching game is a cooperative game defined by a graph $G = (V, E)$. The player set is V and the value of a coalition $S \subseteq V$ is defined as the size of a maximum matching in the subgraph induced by S. We show that the nucleolus of such games can be computed efficiently. The result is based on an alternative characterization of the least core which may be of independent interest. The general case of weighted matching games remains unsolved.

1. INTRODUCTION

A *cooperative game* is defined by a set N of *players* and a *value function* $v : 2^N \to \mathbb{R}$, associating a value $v(S)$ to every subset (*coalition*) $S \subseteq N$. We assume that $v(\emptyset) = 0$. The value $v(S)$ of a coalition $S \subseteq N$ is interpreted as the total gain the members of S can achieve by cooperating.

The central problem in cooperative game theory is how to allocate the total gain $v^* = v(N)$ among the individual players $i \in N$ in a "fair" way. There are various notions of fairness and corresponding allocation rules (*solution concepts*).

Clearly, a useful solution concept should not only be "fair" in an adequate sense but also efficiently computable. The computational complexity of - by now classical - solution concepts has therefore been studied with growing interest during the last years (see, *e.g.*, Deng and Papadimitriou [1994], Granot and Granot [1992], Granot *et al.* [1996], Faigle *et al.* [1997],[1998a], Faigle, Kern and Kuipers [1998b], Deng, Ibaraki and Nagamochi [1999]).

The most prominent and widely accepted solution concept is the *core* of a game:

$$\operatorname{core}(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v^*, x(S) \geq v(S) \text{ for all } S \subsetneq N\}.$$

Here, we use the shorthand notation

$$x(S) := \sum_{i \in S} x_i$$

for $S \subseteq N$. Any $x \in \mathbb{R}^N$ with $x(N) = v^*$ is an *allocation*. So a core allocation $x \in \mathbb{R}^V$ guarantees each coalition $S \subseteq N$ to be satisfied in the sense that it gets at least what it could gain on its own.

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If the core is empty (and even in case it is not) one might try to find allocations x in the *least core*, satisfying all coalitions $S \subsetneq N$ as much as possible. To this end we let $F_0 := \{\emptyset, N\}$ and consider the LP

$$\begin{array}{llrcll} (P_1) & \max & \epsilon & & & \\ & s.t. & x(S) & \geq & v(S)+\epsilon & (S \notin F_0) \\ & & x(N) & = & v^* & \end{array}$$

with optimum value $\epsilon_1 \in \mathbb{R}$. (Clearly, $\epsilon_1 \geq 0$ if and only if core$(N, v) \neq \emptyset$.)

We let $P_1(\epsilon)$ denote the set of all $x \in \mathbb{R}^V$ such that (x, ϵ) satisfies the constraints of (P_1). So core(N, v)= $P_1(0)$. The *least core* is defined as

$$\text{leastcore}(N, v) := P_1(\epsilon_1).$$

The *excess* of a coalition $\emptyset \neq S \subsetneq N$ with respect to an allocation $x \in \mathbb{R}^V$ is defined as

$$e(S, x) := x(S) - v(S).$$

So least core allocations are those that maximize the minimal excess. If the least core is not yet a single point, one might try to find "the best" allocation in the least core by further pursuing the idea of maximizing minimum excess: Given an allocation $x \in \mathbb{R}^V$ define the *excess vector* $\theta(x)$ to be the $2^N - 2$ dimensional vector whose components are the excesses $e(S, x)$, $\emptyset \neq S \subsetneq N$, ordered nondecreasingly. The *nucleolus* (Schmeidler [1969]) is then the (unique!) allocation $x \in \mathbb{R}^V$ that lexicographically maximizes $\theta(x)$.

Although computational aspects shall be discussed later, it is immediately clear that computing the nucleolus by explicit lexicographic optimization of the excess vector is infeasible: In general there are exponentially (in $|N|$) many different excess values, whereas an efficient procedure should be polynomial in $|N|$. The standard procedure for computing the nucleolus proceeds by solving up to $|N|$ linear programs. To present it we introduce the following notation: For a polyhedron $P \subseteq \mathbb{R}^N$ let

$$Fix\, P := \{S \subseteq N \mid x(S) = y(S) \text{ for all } x, y \in P\}$$

denote the set of coalitions *fixed* by P.

Now, assume we have determined the least core $P_1(\epsilon_1)$. We then proceed to maximize the minimal excess on those coalitions which are not already fixed, *i.e.*, we solve

$$\begin{array}{llrcll} (P_2) & \max & \epsilon & & & \\ & s.t. & x & \in & P_1(\epsilon_1) & \\ & & x(S) & \geq & v(S)+\epsilon & S \notin Fix\, P_1(\epsilon_1) \end{array}$$

and let $\epsilon_2 > \epsilon_1$ be the corresponding optimum value. Extending our previous notation in the obvious way, we let $P_2(\epsilon)$ denote the set of all $x \in \mathbb{R}^N$ satisfying the constraints of (P_2) for $\epsilon \in \mathbb{R}$. Now proceed to

$$
\begin{array}{llrcll}
(P_3) & \max & \epsilon & & & \\
& s.t. & x & \in & P_2(\epsilon_2) & \\
& & x(S) & \geq & v(S) + \epsilon & (S \notin Fix\, P_2(\epsilon_2))
\end{array}
$$

etc. until

$$
\begin{array}{llrcll}
(P_r) & \max & \epsilon & & & \\
& s.t. & x & \in & P_{r-1}(\epsilon_{r-1}) & \\
& & x(S) & \geq & v(S) + \epsilon & (S \notin Fix\, P_{r-1}(\epsilon_{r-1}))
\end{array}
$$

defines a unique solution $x^* \in \mathbb{R}^V$, the nucleolus of the game.

Since the feasible regions of the above sequence of LP's decrease in dimension, we conclude that $r \leq |N|$. So we compute at most $|N|$ different excess values explicitly. Note, however, that in each step we have to identify the set $Fix\, P_i(\epsilon_i)$. Furthermore, the number of constraints in each (P_i) remains exponential in $|N|$.

The above "Linear Programming approach" to the nucleolus is also interesting from a structural point of view, as it implies a nice bound on the size $< x^* >$ of the nucleolus (number of bits necessary to represent x^*). Let $< v >$ denote the maximum size of the v-values, *i.e.*, $< v > := \max\{< v(S) > \ | \ S \subseteq N\}$.

Theorem 1.1. *The nucleolus of (N, v) has size bounded polynomially in $|N|$ and $< v >$.*

Proof: Let $F_0 \subset \ldots \subset F_{r-1} \subseteq 2^N$ denote the increasing sequence of fixed sets in $(P_1), \ldots, (P_r)$, *i.e.*, $F_0 = \{\emptyset, N\}$ and

$$F_i := Fix\, P_i(\epsilon_i) \qquad (i = 1, \ldots, r-1).$$

Then the unique lexicographic optimum of

$$
\begin{array}{lrcll}
\text{lex-max } (\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_r, x_1, \ldots, x_{|N|}) & & & & \\
s.t. \quad x(N) & = & v^* & & \\
x(S) & \geq & v(S) + \tilde{\epsilon}_1 & (S \notin F_0) & \\
x(S) & \geq & v(S) + \tilde{\epsilon}_2 & (S \notin F_1) & \\
& \vdots & & & \\
x(S) & \geq & v(S) + \tilde{\epsilon}_r & (S \notin F_{r-1}) &
\end{array}
$$

equals $(\epsilon_1, \ldots, \epsilon_r, x^*)$, where x^* is the nucleolus and $\epsilon_1, \ldots, \epsilon_r$ are the optimum values of $(P_1), \ldots, (P_r)$. Hence $(\epsilon_1, \ldots, \epsilon_r, x^*)$ is a vertex of the feasibility region of the above program. As such its size is polynomial in the dimension $r + |N| = O(|N|)$ and the maximum size of a constraint (*i.e.*, the *facet complexity*, *cf.* Grötschel, Lovász and Schrijver [1988]). The latter is bounded by $|N| + < v >$.

$\diamond$

As to complexity issues in cooperative game theory, various results have been obtained for particular classes of games and solution concepts. For example, so-called minimum spanning tree games have been studied with respect to core, least core and nucleolus, *cf.* Meggido [1987], Granot *et al.* [1996], Faigle *et al.* [1997], Faigle, Kern and Paulusma [1999]. Deng, Ibaraki and Nagamochi [1999] analyze the core of various combinatorial games with respect to complexity. Granot, Granot and Zhu [1998] study the complexity of the nucleolus in general.

The present paper deals with so-called *matching games* (*cf.* section 2) which have been studied already earlier. Solymosi and Raghavan [1994] present an efficient algorithm for computing the nucleolus in the bipartite case (so-called *assignment games*). Deng, Ibaraki and Nagamochi [1999] characterize when the core is empty (*cf.* also section 2). Faigle *et al.* [1998a] introduce the *nucleon* as an alternative to the nucleolus, present an efficient algorithm for computing the nucleon and point out that the problem of computing the nucleolus remains unsolved. Faigle, Kern and Kuipers [1998c] prove a general result on the complexity of the so-called *kernel* (a subset of the least core) of a game. As a consequence of this, computing an element in the least core is easy for matching games. The complexity of the nucleolus remains unsolved yet. In the current paper we solve the "unweighted case" by presenting an efficient algorithm for computing the nucleolus of *cardinality matching games*. Our result is based on a polynomial description of the least core of such games, which might be of independent interest.

2. Matching Games

Let $G = (V, E)$ be a graph and $w : E \to \mathbb{R}^+$ an edge weighting. We use the following standard notation: For $S \subseteq V$ we let $E(S) \subseteq E$ denote the set of edges joining vertices of S. For $F \subseteq E$ we let $V(F)$ denote the set of vertices covered by F. G and w define a cooperative game with player set V. The value of a coalition $S \subseteq V$ is given by

$$v(S) := \max\{w(M) \mid M \subseteq E(S) \text{ is a matching }\},$$

the maximum weight matching in the subgraph induced by S.

In the following we restrict ourselves to *cardinality matching games*. These arise when $w \equiv 1$, *i.e.*, the value function is given by

$$v(S) := \max\{\,|M| \mid M \subseteq E(S) \text{ is a matching }\}.$$

Example

(i) Let $G = K_2$ be the complete graph on two nodes. Then (P_1) has a unique optimal solution, given by the nucleolus $x^* = (\frac{1}{2}, \frac{1}{2})$ and $\epsilon_1 = \frac{1}{2}$.

In the following we assume that $G \neq K_2$.

(ii) Let $G = (V, E)$ be the graph as shown in Figure 2.1. V is split into $\{a\} \cup D_1 \cup D_2$. Then (P_1) has a unique optimal solution: the nucleolus x^* given by

$$x_i^* = \begin{cases} \frac{4}{7} & \text{if } i = a \\ \frac{3}{7} & \text{if } i \in D_1 \cup D_2 \end{cases}$$

and $\epsilon_1 = -\frac{3}{7}$.

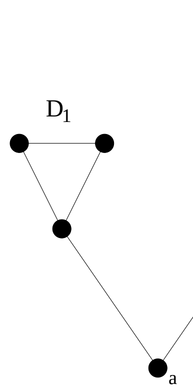

Figure 2.1

(iii) Let $G = (V, E)$ be the graph as shown in Figure 2.2. We have $V = D_1 \cup D_2 \cup D_3$. Then $\epsilon_1 = -1$ and $P_1(-1)$ contains all allocations $x \in \mathbb{R}^{11}$ for which

$$\begin{array}{lll} x_i & = & x_j \quad (i, j \in D_p,\ p = 1, \ldots, 3) \\ x_i + x_j & = & \frac{1}{2} \quad (i \in D_1,\ j \in D_2) \\ x_i & = & \frac{1}{2} \quad (i \in D_3) \\ x & \geq & 0. \end{array}$$

The nucleolus x^* is given by

$$\begin{array}{lll} x^* & \equiv & \frac{1}{4} \quad \text{on } D_1 \cup D_2 \\ x^* & \equiv & \frac{1}{2} \quad \text{on } D_3. \end{array}$$

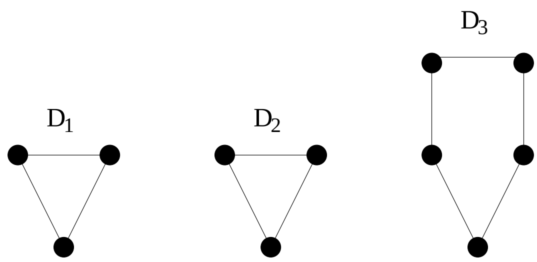

Figure 2.2

In the following we shall need some fundamental results and concepts from matching theory: A (*near-*) *perfect* matching is one that covers all nodes (except one). A graph is *factor-critical* if removing any point results in a perfectly matchable graph.

If $A \subseteq V$, we let as usual $G \backslash A$ denote the graph obtained by removing A. A component of $G \backslash A$ is called *even* or *odd* if it has an even respectively odd number of nodes. We let $C = C(A)$ denote the set of even components of $G \backslash A$ and $D =$

$\mathcal{D}(A)$ the set of odd components of $G\backslash A$. Recall that $A \subseteq V$ is called a *Tutte set* if each maximum matching M of G decomposes as

$$M = M_{\mathcal{C}} \cup M_{A,\mathcal{D}} \cup M_{\mathcal{D}},$$

where $M_{\mathcal{C}}$ is a perfect matching in $\bigcup \mathcal{C}$, the union of all even components. $M_{\mathcal{D}}$ induces a near-perfect matching in all odd components $D \in \mathcal{D}$ and $M_{A,\mathcal{D}}$ is a matching which matches A (completely) into $\bigcup \mathcal{D}$, the union of odd components. Equivalently, A is a Tutte-set if and only if the size υ^* of a maximum matching in G equals

$$\upsilon^* = \sum_{C \in \mathcal{C}} \frac{|C|}{2} + |A| + \sum_{D \in \mathcal{D}} \frac{(|D|-1)}{2}.$$

Tutte sets can be found efficiently. More precisely, the following is true (see, *e.g.*, Lovász and Plummer [1986])

Theorem 2.1 (Gallai-Edmonds Decomposition)**.** *Given $G = (V, E)$ one can efficiently construct a Tutte set $A \subseteq V$ such that*

(i) *all odd components $D \in \mathcal{D}$ are factor-critical*
(ii) *for each $D \in \mathcal{D}$ there is some maximum matching which does not completely cover D.*

In the following we assume that $A \subseteq V$ is a (fixed) Tutte set satisfying the conditions (i) and (ii) in Theorem 2.1. We let $\mathcal{M}^*$ denote the set of maximum matchings in G. Each $M \in \mathcal{M}^*$ matches A completely in $\mathcal{D}$. By condition (ii) of Theorem 2.1, given $D \in \mathcal{D}$ there is some $M \in \mathcal{M}^*$ matching A into $\mathcal{D}\backslash\{D\}$. We say that M *leaves D uncovered.*

We will sometimes identify subsets of V with the corresponding induced subgraphs. For example, if $i \in V$ is a vertex we do not hesitate to write $i \in D$ to indicate that i is a vertex of the component $D \in \mathcal{D}$. If $x \in \mathbb{R}^V$ is an allocation, we consequently write

$$x(D) = \sum_{i \in D} x(i).$$

Finally, we also extend our general shorthand notation in the following way, if no misunderstanding is possible: If $e = (i, j) \in E$, we write $x(e) = x(\{i, j\})$. More generally, if $M \subseteq E$ is a matching, we let $x(M) := x(V(M))$.

After these preliminaries let us study *core(G)* and *leastcore(G)*, the core and least core of the matching game defined by G. We start with the following simple observation (*cf.* also Deng, Ibaraki and Nagamochi [1999]):

Theorem 2.2. *The matching game defined by $G = (V, E)$ has non-empty core ($\epsilon_1 \geq 0$) if and only if $|D| = 1$ for all $D \in \mathcal{D}$.*

Proof: "$\Leftarrow$": Suppose $|D| = 1$ for all $D \in \mathcal{D}$. Then $x \in \mathbb{R}^V$ defined by

$$\begin{array}{lcll} x & \equiv & \frac{1}{2} & \text{on} \bigcup \mathcal{C} \\ x & \equiv & 1 & \text{on } A \\ x & \equiv & 0 & \text{on } \bigcup \mathcal{D} \end{array}$$

is easily seen to be in the core.

"⇒": Suppose $D \in \mathcal{D}$ with $|D| \geq 3$. Let $e = (i, j) \in E(D)$. From Theorem 2.1 we conclude that $G\backslash i$ and $G\backslash j$ have matchings of size v^*. So if $x \in \mathbb{R}^V$ were in the core, then

$$x(V\backslash i) \geq v(V\backslash i) = v^* \text{ and } x(V\backslash j) \geq v(V\backslash j) = v^*.$$

Furthermore, $x(e) = x(\{i, j\}) \geq 1$. Together, these imply $x(V) > v^*$, a contradiction. Hence the core must be empty.

◇

Since the Gallai-Edmonds decomposition can be computed efficiently, we can easily check whether the core is empty or not. In the latter case, the least core and the nucleolus are straightforward to compute. This is essentially due to the fact that all ϵ_i are non-negative:

Theorem 2.3. *In case of non-empty core ($\epsilon_1 \geq 0$) the least core equals the set of allocations $x \in \mathbb{R}^V$ solving*

$$\begin{array}{lllll} (P_1^+) & \max & \epsilon & & \\ & s.t. & x(e) \geq & 1+\epsilon & \textit{for all } e \in E \\ & & x_i \geq & \epsilon & \textit{for all } i \in V \\ & & x(V) = & v^*. & \end{array}$$

Proof: The proof is straightforward, using the fact that the above constraints (for $\epsilon \geq 0$) imply $x(S) \geq v(S) + \epsilon$ for all $S \subsetneq N$.

◇

Remark 2.1. *Note that the optimum value $\epsilon_1^+ = \epsilon_1$ of the LP in Theorem 2.3 is always at most* 0. *(Recall that we assume that $G \neq K_2$, in which case $\epsilon_1 = \frac{1}{2}$.)*

Continuing in a similar way, also the nucleolus can be computed easily. We first identify

$$E_1 := \{e \in E \mid e \in Fix\, P_1^+(\epsilon_1^+)\} \text{ and } V_1 := \{i \in V \mid i \in Fix\, P_1^+(\epsilon_1^+)\}$$

and then solve

$$\begin{array}{lllll} (P_2^+) & \max & \epsilon & & \\ & s.t. & x(e) = & 1+\epsilon_1^+ & (e \in E_1) \\ & & x_i = & \epsilon_1^+ & (i \in V_1) \\ & & x(e) \geq & 1+\epsilon & (e \in E\backslash E_1) \\ & & x_i \geq & \epsilon & (i \in V\backslash V_1) \\ & & x(V) = & v^*. & \end{array}$$

with optimum value $\epsilon_2^+ = \epsilon_2$ etc.

Remark 2.2. *Note that also for general weighted matching games with non-empty core, a similar characterization of the (least) core and nucleolus exists.*

The above approach fails in case $\epsilon_1 < 0$. In this case, at least intuitively, large coalitions $S \subseteq V$ get fixed in the first place rather than small ones (single nodes and edges) as above. The case $\epsilon_1 < 0$ (empty core) is treated in section 3.

3. When the core is empty

In the following we assume that the core is empty. Equivalently, $\epsilon_1 < 0$ and $|D| > 1$ for some odd component $D \in \mathcal{D}$. We first state the following simple fact (which in the non-empty core case follows trivially from Theorem 2.3):

Lemma 3.1. *leastcore(G)* $\subseteq \mathbb{R}_+^V$.

Proof: Assume to the contrary that (x, ϵ_1) is an optimal solution of

$$\begin{array}{llrcll} (P_1) & \max & \epsilon & & & \\ & s.t. & x(S) & \geq & v(S) + \epsilon & (S \notin \mathcal{F}_0) \\ & & x(V) & = & v^* & \end{array}$$

and $x_i < 0$ for some $i \in V$.

Claim: If $S \subseteq V$ satisfies $x(S) \geq v(S) + \epsilon_1$ with equality, then $i \in S$.

Proof: Assume to the contrary that $i \notin S$.
case 1. $S \subset S \cup i \subset V$.
Then $x(S \cup i) < x(S) = v(S) + \epsilon_1 \leq v(S \cup i) + \epsilon_1$ contradicts the feasibility of x.

case 2. $S \subset S \cup i = V$.
Then $x(V) = x(S) + x_i = v(S) + \epsilon_1 + x_i < v(S) \leq v^*$ again contradicts the feasibility of x.

Hence the claim is true. But then we may slightly increase x on $\{i\}$ and decrease x on $V \backslash i$ uniformly by the same total amount, thereby obtaining a better solution. This proves the lemma.

$\diamond$

Due to Lemma 3.1 the problem (P_1) defining the least core can equivalently be stated as follows. Let $\mathcal{M}$ denote the set of matchings $M \subseteq E$. Then (recall our notation $x(M) = x(V(M))$ from section 2):

$$\begin{array}{llrcll} (P_1) & \max & \epsilon & & & \\ & s.t. & x(M) & \geq & |M| + \epsilon & (M \in \mathcal{M}) \\ & & x(V) & = & v^* & \\ & & x & \geq & 0. & \end{array}$$

Proposition 3.1. *Checking whether a given $x \in \mathbb{R}^V$ is an element of $P_1(\epsilon)$ can be done in polynomial time.*

Proof: It suffices to show that for given $x \in \mathbb{R}^V$ and $\epsilon \in \mathbb{R}$ we can sufficiently check whether

$$x(M) \geq |M| + \epsilon \quad (M \in \mathcal{M})$$

holds. This can be done by solving a minimum weight matching problem on $G = (V, E)$ with respect to the edge weights

$$w_{ij} := x_i + x_j - 1 \quad ((i, j) \in E),$$

see, *e.g.*, Lovász and Plummer [1986].

$\diamond$

As a consequence of Proposition 3.1 we can solve (P_1) efficiently (*cf.* Faigle, Kern and Kuipers [1998c] for more detail.) Here we aim for more, namely a concise description of $P_1(\epsilon_1)$.

As a first step we introduce a relaxation $(\hat{P}_1)$ of (P_1) below which is easier to analyze and, as we shall see, defines the same optimum value. To motivate this approach, note that, as mentioned earlier, we expect rather large matchings to become tight when solving (P_1). Let $\mathcal{M}^*$ denote the set of maximum matchings in G and let $\mathcal{M}_D$ denote the set of matchings $M \subseteq E(\bigcup \mathcal{D})$ which are completely contained in the union of the odd components.

We shall study the following relaxation of (P_1):

$$\begin{array}{llrcll} (\hat{P}_1) & \max & \epsilon & & & \\ & s.t. & x(M) & \geq & |M| + \epsilon & (M \in \mathcal{M}^* \cup \mathcal{M}_D) \\ & & x(V) & = & v^* & \\ & & x & \geq & 0 & \end{array}$$

with optimum value $\hat{\epsilon}_1$. (As in the proof of Theorem 2.2, it is easy to see that $\hat{\epsilon}_1 < 0$, *cf.* also below.)

To investigate the structure of optimal solutions of $(\hat{P}_1)$, let us introduce some notation. As before, $\hat{P}_1(\epsilon)$ denotes the set of $x \in \mathbb{R}^V$ such that (x, ϵ) is feasible for $(\hat{P}_1)$. If $x \in \hat{P}_1(\hat{\epsilon}_1)$ is an optimal solution, we say that $M \in \mathcal{M}^* \cup \mathcal{M}_D$ is *x-tight*, if $x(M) = |M| + \hat{\epsilon}_1$. Given a feasible solution $x \in \hat{P}_1(\epsilon)$ and $D \in \mathcal{D}$, let

$$x_D := \frac{x(D)}{|D|}$$

denote the average value of x on D. Define $\bar{x} \in \mathbb{R}^V$ by *averaging* x on each component $D \in \mathcal{D}$, *i.e.*,

$$\bar{x}_i := x_D \qquad (i \in D, D \in \mathcal{D})$$

and leaving x unchanged on $A \cup \bigcup \mathcal{C}$.

Lemma 3.2. *If $x \in \hat{P}_1(\epsilon)$ then $\bar{x} \in \hat{P}_1(\epsilon)$ and $\epsilon < 0$.*

Proof: Let $x \in \hat{P}_1(\epsilon)$. It suffices to show that averaging x on some component $D \in \mathcal{D}$ preserves feasibility. Thus let $D \in \mathcal{D}$ and let $\tilde{x} \in \mathbb{R}^V$ be obtained by averaging x on D, *i.e.*, $\tilde{x}_i = x_D \quad (i \in D)$.

Certainly $\tilde{x}$ satisfies $\tilde{x} \geq 0$ and $\tilde{x}(V) = v^*$. We are left to check $\tilde{x}(M) \geq |M| + \epsilon$ for $M \in \mathcal{M}^* \cup \mathcal{M}_D$.

Suppose $M \in \mathcal{M}^*$. Then either M covers D or $M \cap D = D \backslash i$ for some $i \in D$. In the first case $\tilde{x}(M) = x(M)$ and the claim follows. In the second case we may assume without loss of generality that $i \in D$ maximizes x_i over D, otherwise we replace M inside D by some other near-perfect matching without changing $\tilde{x}(M) - |M|$. (Recall that D is factor-critical.) But then $x_i \geq \tilde{x}_i$ and consequently $\tilde{x}(M) \geq x(M)$, so the claim follows as $x \in \hat{P}_1(\epsilon)$.

Next consider $M \in \mathcal{M}_D$ and assume M minimizes $\tilde{x}(M) - |M|$ over $\mathcal{M}_D$. If $\tilde{x} \equiv x_D > \frac{1}{2}$ on D, then $M \cap D = \emptyset$. Hence $\tilde{x}(M) = x(M)$ and the claim follows. If $\tilde{x} \equiv x_D \leq \frac{1}{2}$ on D, then $M \cap D$ is without loss of generality a near-perfect matching in D and we argue as we did for $M \in \mathcal{M}^*$.

Finally, let us show that $\epsilon < 0$. Let $D \in \mathcal{D}$ be a component with $|D| > 1$. If $\bar{x} \equiv 0$ on D, then $\epsilon \leq -1$. (Indeed, if $e \in E(D)$, then $0 = \bar{x}(e) \geq 1 + \epsilon$.) If $\bar{x} \equiv x_D > 0$ on D, let $M \in \mathcal{M}^*$ be a maximum matching leaving some $i \in D$ unmatched. Then $\bar{x}(M) \geq v^* + \epsilon$ and $\bar{x}(M) < \bar{x}(M \cup i) \leq \bar{x}(V) = v^*$. Hence $\epsilon < 0$.

$\diamond$

We conclude that $\hat{\epsilon}_1 < 0$. If $x \in \hat{P}_1(\hat{\epsilon}_1)$ is an optimal allocation, so is $\bar{x}$. Furthermore, some matchings in $\mathcal{M}^* \cup \mathcal{M}_D$ must be $\bar{x}$-tight. These can in principle be found by minimizing $\bar{x}(M) - |M|$ over $\mathcal{M}^* \cup \mathcal{M}_D$. Minimizing $\bar{x}(M) - |M|$ over $\mathcal{M}^*$ amounts to solving a minimum weight maximum matching problem. Minimizing $\bar{x}(M) - |M|$ over $\mathcal{M}_D$ is even trivial: We simply choose a near-perfect matching in each component $D \in \mathcal{D}$ with $\bar{x} \equiv x_D < \frac{1}{2}$ (plus an arbitrary matching in all components on which $\bar{x} = \frac{1}{2}$). So computing an $\bar{x}$-tight $M \in \mathcal{M}^* \cup \mathcal{M}_D$ for given $\bar{x} \in \hat{P}_1(\hat{\epsilon}_1)$ is easy.

We aim at a more structural characterization of $\bar{x}$-tight matchings for given $\bar{x} \in \hat{P}_1(\hat{\epsilon}_1)$. Let $\mathcal{D}_{max} = \mathcal{D}_{max}(\bar{x}) \subseteq \mathcal{D}$ be the set of odd components on which $\bar{x} \equiv x_D$ is maximum (among all $D \in \mathcal{D}$).

Lemma 3.3. *No $\bar{x}$-tight $M \in \mathcal{M}^*$ covers all $D \in \mathcal{D}_{max}$. If $\bar{x}$-tight matchings in $\mathcal{M}^*$ exist at all, then for each $D \in \mathcal{D}_{max}$ there is some $\bar{x}$-tight $M \in \mathcal{M}^*$ leaving D uncovered.*

Proof: Suppose $\bar{M} \in \mathcal{M}^*$ is $\bar{x}$-tight and covers $D \in \mathcal{D}_{max}$. Let $\tilde{M} \in \mathcal{M}^*$ be any matching not covering D. (Recall Theorem 2.1.) Let $P \subseteq \bar{M} \cup \tilde{M}$ be the unique maximal alternating path starting in D (in a node uncovered by $\tilde{M}$) and ending in, say, $\tilde{D}$ (in a node uncovered by $\bar{M}$). Reversing $\bar{M}$ along P results in a matching $M \in \mathcal{M}^*$ covering $\tilde{D}$ instead of D. Since $D \in \mathcal{D}_{max}$, we have $\bar{x}_D \geq \bar{x}_{\tilde{D}}$, hence $\bar{x}(M) \leq \bar{x}(\bar{M})$.

Thus M must be $\bar{x}$-tight again, proving the second claim. The first claim follows by observing that if $\bar{M}$ would cover all $D \in \mathcal{D}_{max}$, then $\tilde{D} \notin \mathcal{D}_{max}$ (as it is uncovered by $\bar{M}$). But then $\bar{x}_D > \bar{x}_{\tilde{D}}$ and $\bar{x}(M) < \bar{x}(\bar{M})$, hence

$$\bar{x}(M) < \bar{x}(\bar{M}) = |\bar{M}| + \hat{\epsilon}_1 = |M| + \hat{\epsilon}_1,$$

contradicting $\bar{x} \in \hat{P}_1(\hat{\epsilon}_1)$.

$\diamond$

Let M_D^* denote the set of all maximum matchings in M_D.

Lemma 3.4. *Let* $x \in \hat{P}_1(\hat{\epsilon}_1)$. *Then*

(i) $x = \bar{x}$
(ii) $x \le \frac{1}{2}$ *on* $\bigcup D$
(iii) *Each* $M \in M_D^*$ *is x-tight.*

Proof: Let $x \in \hat{P}_1(\hat{\epsilon}_1)$. We first prove (ii) and (iii) for $\bar{x}$ and then show that $x = \bar{x}$.

(ii) $\bar{x} \le \frac{1}{2}$ on $\bigcup D$:
Suppose to the contrary that $\bar{x} > \frac{1}{2}$ on $D \in D_{max}$.

We first consider the case $A \cup \bigcup C = \emptyset$. If $D_{max} = D$, we had $\bar{x} > \frac{1}{2}$ on $\bigcup D$ and hence $\bar{x}(V) > v^*$, a contradiction. Hence $D_{max} \subsetneq D$. Then we may decrease $\bar{x}$ slightly and uniformly on $\bigcup D_{max}$ and increase $\bar{x}$ on $\bigcup D \setminus \bigcup D_{max}$ resulting in some $\bar{\bar{x}} \in \hat{P}_1(\hat{\epsilon}_1)$ for which no $M \in M^* \cup M_D$ is tight. This contradicts the optimality of $\hat{\epsilon}_1$.

Now suppose $A \cup \bigcup C \neq \emptyset$. If $D_{max} = D$, we had $\bar{x} > \frac{1}{2}$ on $\bigcup D$ and hence no $M \in M_D$ were $\bar{x}$-tight. We may thus decrease $\bar{x}$ on $\bigcup D$ and increase $\bar{x}$ on $A \cup \bigcup C$ by the same (sufficiently small) amount $\delta > 0$ resulting in some $\bar{x}^\delta \in \hat{P}_1(\hat{\epsilon}_1)$ for which no $M \in M^* \cup M_D$ is tight. (Recall Lemma 3.3.) This contradicts the optimality of $\hat{\epsilon}_1$.

If $D_{max} \subsetneq D$, we proceed similarly. Chose $\delta > 0$ sufficiently small and let $\bar{x}^\delta$ arise from $\bar{x}$ by

- decreasing $\bar{x}_i$ by $\dfrac{\delta}{|D|}$ $\quad (i \in D, D \in D_{max})$
- increasing $\bar{x}$ on A by δ' in total, where $(|D_{max}| - 1)\delta < \delta' < |D_{max}|\delta$
- increasing $\bar{x}$ uniformly on $\bigcup D \setminus \bigcup D_{\max}$ by $|D_{max}|\delta - \delta'$ in total.

For sufficiently small $\delta > 0$ the resulting $\bar{x}^\delta$ has $\bar{x}^\delta(M) > \bar{x}(M)$ for each $\bar{x}$-tight $M \in M_D$ (since none of these meets D_{max}) and $\bar{x}^\delta(M) > \bar{x}(M)$ for all $\bar{x}$-tight $M \in M^*$ by Lemma 3.3. Hence, again $\bar{x}^\delta \in \hat{P}_1(\hat{\epsilon}_1)$ has no tight matchings, contradicting the optimality of $\hat{\epsilon}_1$.

(iii) Each $M \in M_D^*$ is $\bar{x}$-tight: Since $\bar{x} \le \frac{1}{2}$ on $\bigcup D$, each $M \in M_D^*$ minimizes $\bar{x}(M) - |M|$ over M_D. It therefore suffices to show that *some* matching in M_D is $\bar{x}$-tight. Assume to the contrary that no matching in M_D is $\bar{x}$-tight. As above, this exludes $M^* \subseteq M_D$, so $A \cup \bigcup C$ must be non-empty.

case 1. $\bar{x}(\bigcup D) > 0$.
In this case we may slightly (and uniformly) decrease $\bar{x}$ on $\bigcup D_{\max}$ and increase it by the same total amount on $A \cup \bigcup C$. By Lemma 3.3 the resulting $\bar{\bar{x}}$ has no tight matchings, a contradiction.

case 2. $\bar{x} \equiv 0$ on $\bigcup D$.
Then $\bar{x}(M) = v^*$ for $M \in M^*$. Since $\hat{\epsilon}_1 < 0$, no $M \in M^*$ were tight either, a contradiction.

(i) $x = \bar{x}$:
For each $D \in \mathcal{D}$ we chose a node $i \in D$ with maximum x-value and a near-perfect matching covering $D \backslash i$. Let $M \in \mathcal{M}_D^*$ be the union of all these near-perfect matchings. By construction we have $x(M) \leq \bar{x}(M)$ with equality if and only if $x \equiv \bar{x}$ on $\bigcup \mathcal{D}$. But since M is $\bar{x}$-tight,

$$x(M) < \bar{x}(M) = |M| + \hat{\epsilon}_1$$

would contradict $x \in \hat{P}_1(\hat{\epsilon}_1)$.

$\diamond$

Lemma 3.5. *Let $x = \bar{x} \in \hat{P}_1(\hat{\epsilon}_1)$. Then there is some x-tight $M \in \mathcal{M}^*$. Moreover, if $D \in \mathcal{D}_{\max}$ or $|D| > 1$ then there is some x-tight $M \in \mathcal{M}^*$ not covering D.*

Proof: The lemma is trivial in case $A \cup \bigcup \mathcal{C} = \emptyset$. So we suppose $A \bigcup \mathcal{C} \neq \emptyset$ and we first claim that any $x \in \hat{P}_1(\hat{\epsilon}_1)$ has $x(A \cup \bigcup \mathcal{C}) > 0$. Indeed, any $M \in \mathcal{M}^*$ decomposes as

$$M = M_C \cup M_{A,D} \cup M_D$$

with M_C a perfect matching of $\bigcup \mathcal{C}$, $M_{A,D}$ matching A into D and $M_D \in \mathcal{M}_D^*$. Since $x \in \hat{P}_1(\hat{\epsilon}_1)$, we have

$$x(M) \geq |M_C| + |M_{A,D}| + |M_D| + \hat{\epsilon}_1.$$

Since M_D is x-tight (*cf.* Lemma 3.4(ii)), we have $x(M_D) = |M_D| + \hat{\epsilon}_1$, hence $x(M_C \cup M_{A,D}) \geq |M_C| + |M_{A,D}|$. Since $x \leq \frac{1}{2}$ on $\bigcup \mathcal{D}$, we conclude that indeed

$$x(A \cup \bigcup \mathcal{C}) = x(A \cup M_C) \geq \frac{|A|}{2} + |M_C| > 0.$$

Now let us show that some $M \in \mathcal{M}^*$ is $\bar{x}$-tight. Suppose to the contrary that $x(M) > |M| + \hat{\epsilon}_1$ for all $M \in \mathcal{M}^*$. We could then decrease (somehow) $\bar{x}$ on $A \cup \bigcup \mathcal{C}$ and increase $\bar{x}$ uniformly on $\bigcup \mathcal{D}$ by the same total (sufficiently small) amount. The resulting $\bar{\bar{x}}$ were still in $\hat{P}_1(\hat{\epsilon}_1)$ and would contradict Lemma 3.4 (iii).

By Lemma 3.3, this implies that each $D \in \mathcal{D}_{\max}$ is uncovered by some $\bar{x}$-tight $M \in \mathcal{M}^*$. We are left to prove a corresponding result for $D \in \mathcal{D}$ with $|D| > 1$. Hence assume $D \in \mathcal{D} \backslash \mathcal{D}_{\max}$ and $|D| > 1$. Then $\bar{x} < \frac{1}{2}$ on D by Lemma 3.4 (ii), so every $\bar{x}$-tight $M \in \mathcal{M}_D^*$ contains a near-perfect matching of D. Now suppose D is covered by every $\bar{x}$-tight $M \in \mathcal{M}^*$. We may then decrease $\bar{x}$ slightly on $A \cup \mathcal{C}$ and increase x uniformly on D by the same (sufficiently small) total amount. The resulting $\bar{\bar{x}}$ would again be in $\hat{P}_1(\hat{\epsilon}_1)$ and contradict Lemma 3.4(iii). This finishes the proof.

$\diamond$

Let us call an allocation $x = \bar{x} \in \hat{P}_1(\hat{\epsilon}_1)$ *flexible* if the conclusion of Lemma 3.5 holds with respect to all $D \in \mathcal{D}$, *i.e.*, if each $D \in \mathcal{D}$ is uncovered by some $\bar{x}$-tight $M \in \mathcal{M}^*$.

Lemma 3.6. *Flexible allocations exist.*

Proof: Let $x = \bar{x} \in P_1(\hat{\epsilon}_1)$. Suppose $\bar{x}$ is not already flexible. Then there exists a component $D = \{i\} \in \mathcal{D}$ of size 1 such that every $\bar{x}$-tight $M \in \mathcal{M}^*$ covers i. In particular, this implies that $A \neq \emptyset$. We may thus increase $\bar{x}_i$ and decrease $\bar{x}$ on A by the same total amount δ until $\bar{x}$ becomes "flexible" with respect to $D = \{i\}$. In other words, we choose $\delta > 0$ maximal such that the modification $\bar{x}^\delta$ is still in $\hat{P}(\hat{\epsilon}_1)$. Then $\bar{x}^\delta(M) = |M| + \hat{\epsilon}_1$ holds for a matching $M \in \mathcal{M}^*$ that does not cover i (and is not $\bar{x}$-tight). Because all matchings in $\mathcal{M}^*$ that were already $\bar{x}$-tight (and contain i) remain tight, the claim follows by induction.

$\diamond$

We are now ready to determine the structure of x-tight matchings in $\mathcal{M}^*$ for flexible $x = \bar{x} \in \hat{P}_1(\hat{\epsilon}_1)$. Suppose $\hat{x} \in \hat{P}(\hat{\epsilon}_1)$ is a given flexible allocation. Suppose that $\alpha_o < \ldots < \alpha_p \ \ (p \geq 0)$ are the different values $\hat{x}$ takes on $\bigcup \mathcal{D}$ and let

$$\mathcal{D} = \mathcal{D}_0 \cup \ldots \cup \mathcal{D}_p$$

be the corresponding partition of $\mathcal{D}$. Hence $\hat{x} \equiv \alpha_i$ on $\bigcup \mathcal{D}_i$ and $\mathcal{D}_p = \mathcal{D}_{\max}$.

Proposition 3.2. *There exists a partition $A = A_0 \cup \ldots \cup A_p$ (with some of the A_i possibly empty) such that $M \in \mathcal{M}^*$ is $\hat{x}$-tight if and only if M matches each A_i into $\mathcal{D}_i$.*

Proof: If $A = \emptyset$, the claim is true in the sense that nothing is matched into $\mathcal{D}$ and each $M \in \mathcal{M}^*$ is $\hat{x}$-tight. (By Lemma 3.5, some $\bar{x}$-tight $M \in \mathcal{M}^*$ exists and since $A = \emptyset$, all $M \in \mathcal{M}^* = \mathcal{M}^*_D$ have the same $\hat{x}$-value.)

In general, recall that $\hat{x}$-tight matchings in $\mathcal{M}^*$ are exactly those that minimize $\hat{x}(M)$ over $\mathcal{M}^*$. For given $\hat{x}$, the value $\hat{x}(M)$ only depends on how many nodes of A are matched into each $\mathcal{D}_i$. (This readily follows from the decomposition $M = M_C \cup M_{A,C} \cup M_D$.) In other words, $\hat{x}(M)$ only depends on the total $\hat{x}$-weight of nodes in $\bigcup \mathcal{D}$ that are matched with A. The claim therefore follows from Lemma 3.7 below.

$\diamond$

Lemma 3.7. *Consider a bipartite graph $G(A, B)$ with node set $A \cup B$. Suppose $B = B_0 \cup \ldots \cup B_p$ is a partition of B and edges incident with B_i have weight α_i $(\alpha_0 < \ldots < \alpha_p)$. Assume that the set $\mathcal{M}^*$ of matchings that completely match A into B is non-empty and let $\mathcal{M}^*_{\min}$ be the set of $M \in \mathcal{M}^*$ with minimum weight. Suppose finally, that $\mathcal{M}^*_{\min}$ is "flexible" in the sense that each $b \in B$ is left unmatched by some $M \in \mathcal{M}^*_{\min}$. Then there is a partition $A = A_0 \cup \ldots \cup A_p$ of A such that $M \in \mathcal{M}^*_{\min}$ if and only if M matches A_i into B_i $(i = 0, \ldots, p)$.*

Proof: Let $\mathcal{M}^*_0$ denote the set of maximum matchings in the subgraph G_0 induced by $A \cup B_0$. Clearly, each $M \in \mathcal{M}^*_{\min}$ induces a maximum matching $M_0 \subseteq M$ in $\mathcal{M}^*_0$. (Apply an augmenting path argument.) Hence we must have

$$(*) \ \ \text{Each } b \in B_0 \text{ is left uncovered by some } M_0 \in \mathcal{M}^*_0.$$

Suppose m^*_0 is the maximum size of a matching in G_0. As G_0 is bipartite, Königs Theorem ensures the existence of a vertex cover $A^*_0 \cup B^*_0 \quad (A^*_0 \subseteq A, B^*_0 \subseteq B)$ of size m^*_0. Each $M \in \mathcal{M}^*_0$ is incident with all nodes in $A^*_0 \cup B^*_0$. Hence, by (*) we conclude that $B_0^* = \emptyset$. In other words, each $M \in \mathcal{M}^*_{\min}$ matches A_0^* into B_0. The claim follows by induction.

$\diamond$

We are now prepared to present our main result, a simple alternative description of the least core. Consider the LP

$$
\begin{array}{llrcll}
(\hat{\hat{P}}_1) & \max & \epsilon & & & \\
 & s.t. & x & \equiv & \bar{x} & \\
 & & x_i & \le & \frac{1}{2} & (i \in \bigcup \mathcal{D}) \\
 & & x(e) & \ge & 1 & (e \in E \backslash E(\bigcup \mathcal{D})) \\
 & & x(V) & = & v^* & \\
 & & x(M) & \ge & |M| + \epsilon & (M \in \mathcal{M}^*_{\mathcal{D}}) \\
 & & x & \ge & 0. &
\end{array}
$$

Note that $x \equiv \bar{x}$ is just a shorthand for a number of linear equalities of the type $x_i = x_j$. Further note that for $x \equiv \bar{x}$ the value $x(M)$ is independent of the particular choice of $M \in \mathcal{M}^*_{\mathcal{D}}$. Hence the exponentially many constraints for $M \in \mathcal{M}^*_{\mathcal{D}}$ reduce to one single inequality.

Again, we let $\hat{\hat{P}}_1(\epsilon) := \{x \mid (x, \epsilon) \text{is feasible for} (\hat{\hat{P}}_1)\}$ and denote the optimum value of $(\hat{\hat{P}}_1)$ by $\hat{\hat{\epsilon}}_1$.

Theorem 3.1. *We have $\epsilon_1 = \hat{\epsilon}_1 = \hat{\hat{\epsilon}}_1$ and $leastcore(G) = P_1(\epsilon_1) = \hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1)$*

Proof:

- We have $\epsilon_1 \le \hat{\epsilon}_1$ by definition.
- $\hat{\epsilon}_1 \le \hat{\hat{\epsilon}}_1$: Let $\hat{x} \in \hat{P}_1(\hat{\epsilon}_1)$ be flexible with corresponding partitions $\mathcal{D} = \mathcal{D}_0 \cup \ldots \cup \mathcal{D}_p$ and $A = A_0 \cup \ldots \cup A_p$. Define $\hat{\hat{x}} \in \mathbb{R}^V$ by

$$
\begin{array}{rcll}
\hat{\hat{x}} & \equiv & \frac{1}{2} & \text{on } \bigcup \mathcal{C} \\
\hat{\hat{x}} & \equiv & \hat{x} & \text{on } \mathcal{D} \\
\hat{\hat{x}} & \equiv & 1 - \alpha_i & \text{on } A_i.
\end{array}
$$

We show that $\hat{\hat{x}} \in \hat{\hat{P}}_1(\hat{\epsilon}_1)$ (proving that $\hat{\hat{\epsilon}}_1 \ge \hat{\epsilon}_1$). The only non-trivial constraints to check are $\hat{\hat{x}}(V) = v^*$ and $\hat{\hat{x}}(e) \ge 1$ for $e \in E \backslash E(\bigcup D)$. All other constraints directly follow from Lemma 3.4.

Let $M \in \mathcal{M}^*$ be $\hat{x}$-tight and decompose it as

$$M = M_C \cup M_{A,D} \cup M_D$$

as usual. Since $M_D \in \mathcal{M}^*_{\mathcal{D}}$ is also $\hat{x}$-tight by Lemma 3.4, we conclude that $\hat{x}(M_C \cup M_{A,D}) = |M_C| + |M_{A,D}| = \hat{\hat{x}}(M_C \cup M_{A,D})$ by definition of $\hat{\hat{x}}$. Hence $\hat{\hat{x}}(V) = \hat{x}(V) = v^*$.

Secondly, let us consider $e \in E \backslash E(\bigcup \mathcal{D})$. If $e \in E(A \cup \bigcup \mathcal{C})$ then $\hat{\hat{x}}(e) \ge 1$ by definition of $\hat{\hat{x}}$. (Recall that $\hat{x} = \alpha_i \le \frac{1}{2}$ on $\bigcup \mathcal{D}_i$.) Thus we are left with edges between A and $\bigcup \mathcal{D}$. Suppose $\hat{\hat{x}}(e) < 1$ for such an edge joining, say, $D \in \mathcal{D}_i$ with $a \in A_j$. Then $\hat{\hat{x}}(e) = \alpha_i + 1 - \alpha_j < 1$, *i.e.*, $\alpha_i < \alpha_j$. Since $\hat{x}$ is flexible, there exists an $\hat{x}$-tight matching $M \in \mathcal{M}^*$ not covering D. Since D is factor-critical (and $\hat{x}$ is constant on D), we may assume that M does not match the endpoint of e in D. Since M is $\hat{x}$-tight, $a \in A_j$ is matched into D_j by some edge $f \in M$ (*cf.* Proposition 3.2). But then $M' = M \setminus f + e$ has $\hat{x}(M') < \hat{x}(M)$, a contradiction.

- $\hat{\hat{\epsilon}}_1 \leq \epsilon_1$: We show that in general $\hat{\hat{P}}_1(\epsilon) \subseteq P_1(\epsilon)$. Suppose $x \in \hat{\hat{P}}_1(\epsilon)$. Then $x(M) \geq |M| + \epsilon$ for all $M \in \mathcal{M}_D^*$. Since $x \leq \frac{1}{2}$ on $\bigcup \mathcal{D}$, this also implies $x(M) \geq |M| + \epsilon$ for all $M \in \mathcal{M}_D$. (Use an augmenting path argument.) Since $x(e) \geq 1$ for all $e \in E \setminus E(\bigcup \mathcal{D})$, we further conclude that $x(M) \geq |M| + \epsilon$ for all $M \in \mathcal{M}$.

- Finally, let us verify that $P_1(\epsilon_1) = \hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1)$. We have just proved that "$\supseteq$" holds. Conversely let $x \in P_1(\epsilon_1)$. Then $x \in \hat{P}_1(\hat{\epsilon}_1)$ and, by Lemma 3.4, x satisfies all constraints of $\hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1)$ except possibly $x(e) \geq 1$ for $e \in E \setminus E(\bigcup \mathcal{D})$. Thus let $e \in E \setminus E(\bigcup \mathcal{D})$. Pick $M \in \mathcal{M}_D^*$ not covering the endpoint of e in $\bigcup \mathcal{D}$, so that $M \cup e$ is a matching again. Then, since $x \in P_1(\epsilon_1)$, we have $x(M \cup e) \geq |M| + 1 + \epsilon_1$, and since $M \in \mathcal{M}_D^*$ is x-tight, we have $x(M) = |M| + \hat{\epsilon}_1$. Since $\epsilon_1 = \hat{\epsilon}_1$, the claim follows.

$\diamond$

4. The Nucleolus

Recall from section 1 that the nucleolus is computed by solving the following sequence of LP's:

$$\begin{array}{llrcll} (P_1) & \max & \epsilon & & & \\ & s.t. & x(S) & \geq & v(S) + \epsilon & (S \notin \{\emptyset, V\}) \\ & & x(V) & = & v^* & \end{array}$$

with optimum value ϵ_1,

$$\begin{array}{llrcll} (P_2) & \max & \epsilon & & & \\ & s.t. & x & \in & P_1(\epsilon_1) & \\ & & x(S) & \geq & v(S) + \epsilon & (S \notin Fix\, P_1(\epsilon_1)\ ) \end{array}$$

with optimum value ϵ_2, etc. until the nucleolus is finally determined as the unique solution x^*, $\epsilon^* = \epsilon_r$ of

$$\begin{array}{llrcll} (P_r) & \max & \epsilon & & & \\ & s.t. & x & \in & P_{r-1}(\epsilon_{r-1}) & \\ & & x(S) & \geq & v(S) + \epsilon & (S \notin Fix\, P_{r-1}(\epsilon_{r-1})\ ). \end{array}$$

By Theorem 3.1, (P_1) is equivalent to $(\hat{\hat{P}}_1)$ in the sense that they define the same set of optimal solutions. As we shall see, similar equivalent formulations can be found for (P_k), $\ k \geq 2$. Define recursively

$$
\begin{array}{llcll}
(\hat{\hat{P}}_k) & \max & \epsilon & & \\
& s.t. \quad x & \in & \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1}) & \\
& \quad x(e) & \leq & 1+\epsilon_1-\epsilon & (e \in E(\bigcup \mathcal{D}), e \notin Fix\, \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})\ ) \\
& \quad x(e) & \geq & 1-\epsilon_1+\epsilon & (e \in E\backslash E(\bigcup \mathcal{D}), e \notin Fix\, \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})\ ) \\
& \quad x_i & \geq & -\epsilon_1+\epsilon & (i \in V, i \notin Fix\, \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})\ ).
\end{array}
$$

As before, let $\hat{\hat{\epsilon}}_k$ denote the optimum value of $(\hat{\hat{P}}_k)$ and define $\hat{\hat{P}}_k(\epsilon)$ in the obvious way.

Theorem 4.1. *We have $\epsilon_k = \hat{\hat{\epsilon}}_k$ and $P_k(\epsilon_k) = \hat{\hat{P}}_k(\hat{\hat{\epsilon}}_k)$ for $k = 1, \ldots, r$. In particular, the sequence $\hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1) \supset \ldots \supset \hat{\hat{P}}_r = \{x^*\}$ defines the nucleolus.*

Proof: For $k = 1$ the claim is equivalent to Theorem 3.1. We proceed by induction on k. Assume that $\epsilon_{k-1} = \hat{\hat{\epsilon}}_{k-1}$ and $P_{k-1}(\epsilon_{k-1}) = \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$. The induction step amounts to show the following two things.

(i) $\quad P_k(\epsilon) \subseteq \hat{\hat{P}}_k(\epsilon)$ (implying that $\hat{\hat{\epsilon}}_k \geq \epsilon_k$) :
Let $x \in P_k(\epsilon)$. Then $x \in P_1(\epsilon_1) = \hat{P}_1(\hat{\epsilon}_1)$, so x satisfies $x \geq 0$, $x \equiv x_D \leq \frac{1}{2}$ $(D \in \mathcal{D})$ and $x(M) = |M| + \epsilon_1$ $(M \in \mathcal{M}^*_D)$.

We first consider $e \in E\backslash E(\bigcup \mathcal{D})$ and show that $x(e) \geq 1+\epsilon-\epsilon_1$ unless $e \in Fix\, \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1}) = Fix\, P_{k-1}(\epsilon_{k-1})$. Choose $M \in \mathcal{M}_D$ such that $M \cup e$ is a matching. (Existence follows from the fact that each $D \in \mathcal{D}$ is factor-critical.) Since M is fixed by $\hat{\hat{P}}_1(\hat{\hat{\epsilon}}_1) = P_1(\epsilon_1)$, it is fixed by $\hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$. Hence $e \in Fix\, \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$ if and only if $M \cup e \in Fix\, \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$. Since we assume $e \notin Fix\, \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$, we have $M \cup e \notin Fix\, P_{k-1}(\epsilon_{k-1})$ and thus $x \in P_k(\epsilon)$ implies $x(M \cup e) \geq |M \cup e| + \epsilon$. Together with $x(M) = |M| + \epsilon_1$ this yields $x(e) \geq 1+\epsilon-\epsilon_1$.

In the same way we can show that $x_i \geq \epsilon - \epsilon_1$ for a vertex $i \notin Fix\, \hat{\hat{P}}(\hat{\hat{\epsilon}}_{k-1})$.

Next consider $e \in E(\bigcup \mathcal{D})$, say $e \in E(D)$ for $D \in \mathcal{D}$. We show that $x(e) \leq 1 - \epsilon + \epsilon_1$ unless e is already fixed by $\hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1}) = P_{k-1}(\epsilon_{k-1})$. Since $x \equiv x_D$ on $D \in \mathcal{D}$, we conclude that $x(e)$ is independent of the particular choice of $e \in E(D)$. Choose any $M \in \mathcal{M}^*_D$ and assume without loss of generality that $e \in M \cap E(D)$ is not fixed by $P_{k-1}(\epsilon_{k-1})$. Since $x(M)$ is fixed (to $|M| + \epsilon_1$), we conclude that $M\backslash e \notin Fix\, P_{k-1}(\epsilon_{k-1})$. Hence $x \in P_k(\epsilon)$ implies $x(M\backslash e) \geq |M\backslash e| + \epsilon$. Together with $x(M) = |M| + \epsilon_1$ we get $x(e) \leq 1+\epsilon_1-\epsilon$.

(ii) $\quad \hat{\hat{P}}_k(\epsilon) \subseteq P_k(\epsilon)$ (implying that $\epsilon_k \geq \hat{\hat{\epsilon}}_k$):
Let $x \in \hat{\hat{P}}_k(\epsilon)$. Again this implies $x \in \hat{P}_1(\hat{\epsilon}_1)$, so $x \geq 0$, $x \equiv x_D \leq \frac{1}{2}$ on each $D \in \mathcal{D}$ and $x(M_D) = |M_D| + \epsilon_1$ for $M_D \in \mathcal{M}^*_D$. We are to show that $x(S) \geq v(S) + \epsilon$ for $S \subseteq V$ not yet fixed by $P_{k-1}(\epsilon_{k-1}) = \hat{\hat{P}}_{k-1}(\hat{\hat{\epsilon}}_{k-1})$. Since $x \geq 0$, we may only

consider $S = v(M)$ for $M \in \mathcal{M}$. Furthermore, since $x(e) \geq 1$ on $E \backslash E(\bigcup \mathcal{D})$, we may restrict ourselves to $M \subseteq E(\bigcup \mathcal{D})$. Finally, since $x \equiv x_D \ \ (D \in \mathcal{D})$, $x(M)$ only depends on $|M \cap D|$ for each $D \in \mathcal{D}$. So we may without loss of generality assume that $M \subseteq M_D$ for some $M_D \in \mathcal{M}_D^*$. Assume that M is not fixed by $P_{k-1}(\epsilon_{k-1})$. Since M_D is fixed by $P_{k-1}(\epsilon_{k-1})$, we conclude that $M_D \backslash M$ is not fixed by $P_{k-1}(\epsilon_{k-1})$. So at least some $e \in M_D \backslash M$ is not fixed by $P_{k-1}(\epsilon_{k-1})$. Hence $x \in \hat{\hat{P}}_k(\epsilon)$ implies $x(e) \leq 1 - \epsilon + \epsilon_1$. All other edges $f \in M_D \backslash M$ satisfy $x(f) \leq 1$ (as $x \leq \frac{1}{2}$ on $\bigcup \mathcal{D}$). Hence $x(M_D) = |M_D| + \epsilon_1$ implies $x(M) \geq |M| + \epsilon$ as required.

$\diamond$

Corollary 4.1. *The nucleolus x^* of a matching game on a graph $G = (V, E)$ with unit edge weights can be computed in polynomial time.*

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